SPECTRAL MEASURE FOR THE OPERATOR $R+L$ IN l^2

Let l^2 be the space of sequences $c=(c_0,c_1,\dots)$ such that $\|c\|^2=\sum|c_j|^2<\infty$. Let R and L be the right shift operator and the left shift operator, respectively. Our goal is to compute the spectral measure for the operator $A=L+R$, associated to a vector $e_0=(1,0,0,\dots)$. Notice that e_0 is a cyclic vector for the operator A. Indeed, $A^ne_0=e_n+$a linear combination of $e_0,\dots,e_{n-1}$. Here, e_n is a vector the n-th component of which equals 1, and all other components equal 0. This implies that the span of $e_0,Ae_0,\dots,A^ne_0$ coincides with the span of $e_0,\dots,e_n$, and the span of the vectors A^ne_0, $n=0,1,\dots$, is dense in l^2.

First, let us recall how things look like in a simpler case of $H=l^2(\mathbb{Z})$, the space of doubly infinite sequences. In that case, e_0 is not a cyclic vector, and the closure of the span of A^ne_0, $n=0,1,\dots$, is the subspace of all sequences such that $c_{-k}=c_k$. The operator $U:l^2(\mathbb{Z})\to L^2(S^1)$ given by

$$e_k\mapsto\frac{1}{\sqrt{2\pi}}e^{ikx}$$

is unitary, and $UAU^{-1}=M_{2\cos x}$ where by $M_{f(x)}$ I denote the operator of multiplying by the function $f(x)$. Then

$$(f(A)e_0,e_0)=\frac{1}{2\pi}\int_0^{2\pi}f(2\cos x)dx=\int_{-2}^{2}f(\lambda)\frac{d\lambda}{\pi\sqrt{4-\lambda^2}};$$

the spectral measure is supported on $[-2,2]$, and it equals

$$d\nu=\frac{d\lambda}{\pi\sqrt{4-\lambda^2}}. \tag{1}$$

One can compute the moments of the measure $d\nu$. Notice that $L=R^{-1}$, so

$$A^ne_0=(R+L)^ne_0=\sum_{\sigma_1,\dots,\sigma_n=\pm1}R^{\sigma_n}\cdots R^{\sigma_1}e_0, \tag{2}$$

and

$$(A^ne_0,e_0)=\#\{\sigma_1,\dots,\sigma_n=\pm1:\sum_{j=1}^n\sigma_j=0\}=\begin{cases}0, & \text{when } n \text{ is odd}\\ \binom{2k}{k}, & \text{when } n=2k\end{cases}. \tag{3}$$

Now we come back to the space l^2. The operator L is not an inverse to R anymore; nevertheless, I will write $L=R^{-1}$, keeping in mind that it is not an inverse. Then formula (2) remains valid. Notice that $R^{\sigma_n}\cdots R^{\sigma_1}e_0=0$ when $\sum_{j=1}^l\sigma_j<0$ for some l, $1\le l\le n$. Therefore,

$$(A^ne_0,e_0)=\#\{\sigma_1,\dots,\sigma_n=\pm1:\sum_{j=1}^n\sigma_j=0,\sum_{j=1}^l\sigma_j\ge0 \text{ for } l=1,2,\dots n\}.$$

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This number equals 0 for odd values of n; for $n = 2k$, we denote it by $b(k)$. It can be interpreted as a number of random lattice walks of length $2k$ that start and end at the point 0, and that remain in the non-negative half-line $x \geq 0$ all the time. By $c(k)$ I denote the number of random walks of length $2k$ that start and end at the point 0, and that remain in the positive half-line $x > 0$ at all times except of the initial moment and the final moment. Notice that $c(k) = b(k-1)$. For a walk that contributes to the number $b(k)$, let $2k_1$, $2(k_1 + k_2)$, …, $2(k_1 + k_2 + \cdots k_l) = 2k$ be consecutive moments when it returns to the initial point 0. In terms of σ_j's,

$$\sum_{j=1}^{k_1+\cdots+k_p} \sigma_j = 0, \quad p = 1, \dots, l.$$

The number of walks with fixed $k_1, \dots, k_l$ equals $c(k_1) \cdots c(k_l)$. We conclude that

$$c(k+1) = b(k) = \sum_{k_1+\cdots+k_l=k} c(k_1) \cdots c(k_l). \tag{4}$$

The initial condition is $c(1) = 1$. Let

$$c(t) = \sum_{k=1}^{\infty} c(k) t^k$$

be the generating function for the sequence $c(k)$. Formulas (4), together with $c(1) = 1$, imply

$$\sum_{q=1}^{\infty} c(t)^q = \sum_{k=1}^{\infty} c(k+1) t^k = \frac{c(t) - t}{t}$$

and

$$\frac{c(t)}{1 - c(t)} = \frac{c(t) - t}{t}.$$

One easily solves the last equation for $c(t)$:

$$c(t) = \frac{1 - \sqrt{1 - 4t}}{2}.$$

One expands $c(t)$ into Taylor series

$$c(t) = \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} t^k$$

to get

$$c(k) = \frac{1}{k} \binom{2k-2}{k-1},$$

and, therefore,

$$b(k) = c(k+1) = \frac{1}{k+1} \binom{2k}{k}. \tag{5}$$

These are *Catalan numbers* that are rather ubiquitous (they are denoted usually by C_k.)

Now, I will be looking for a measure, odd moments of which equal 0, and the $2k$-th moment equals $b(k)$. I will look for a measure $w(\lambda)d\lambda$:

$$\int \lambda^{2k} w(\lambda) d\lambda = \frac{1}{k+1}\binom{2k}{k}.$$

Let us do partial integration

$$\int \lambda^{2k} w(\lambda) d\lambda = -\frac{1}{2k+2}\int \lambda^{2k+2}\left(\frac{w(\lambda)}{\lambda}\right)' d\lambda = -\frac{1}{2k+2}\int \lambda^{2k}(\lambda w'(\lambda) - w(\lambda))d\lambda.$$

If one compares this to (1), (3) then one can see that

$$\lambda w'(\lambda) - w(\lambda) = -\frac{2}{\pi\sqrt{4-\lambda^2}}.$$

The even solution of this differential equation (the function $\alpha(\lambda)$ must be even because all odd moments vanish) is

$$w(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}. \tag{6}$$

The measure $d\mu = w(\lambda)d\lambda$ represents a famous *Wigner semicircle law.*