AN EXAMPLE OF A STRONG RESOLVENT CONVERGENT FAMILY OF OPERATORS

Let $H = -d^2/dx^2 + x^2$ be the harmonic oscillator on $L^2(\mathbb{R})$. For $T \geq 1$, I define a potential

$$V_T(x) = \begin{cases} x^2, & \text{when } |x| \leq T \\ 1, & \text{when } |x| > T, \end{cases}$$

and let $H_T = -d^2/dx^2 + V_T(x)$. Operators H and H_T are associated to quadratic forms

$$Q[u] = \int_{-\infty}^{\infty} \big(|u'(x)|^2 + x^2|u(x)|^2\big)dx \text{ and } Q_T[u] = \int_{-\infty}^{\infty} \big(|u'(x)|^2 + V_t(x)|u(x)|^2\big)dx,$$

respectively. Notice that the family of quadratic forms Q_T is increasing as T increases. Therefore, $H_T \geq cI$ for every $T \geq 1$ and $H \geq cI$ where

$$c = \inf\{Q_1[u]/||u||^2\} > 0$$

is the bottom of the spectrum of H_1.

I will show that H_T converges to H as $T \to \infty$ in the strong resolvent sense. Because the operators H_T^{-1} are uniformly bounded, it is sufficient to prove that $H_T^{-1}g \to H^{-1}g$ for functions $g(x)$ that belong to a set that is dense in $L^2(\mathbb{R})$. For this set, I take $C_0^\infty(\mathbb{R})$. It is sufficient to consider real-valued functions: an arbitrary function is a linear combination of its real part and its imaginary part. Now, let $g(x)$ be a real-valued C_0^∞-function, and let $Hf(x) = H_T f_T(x) = g(x)$. Both functions $f(x)$ and $f_T(x)$ are real-valued. The domain of the quadratic form Q_T contains the domain of the quadratic form Q. Therefore, the domain of the operator H is smaller than the domain of the operator H_T. In particular, $f(x)$ belongs to the domain of H_T. One has $g(x) = 0$ and $V_T(x) = 1$ when $|x|$ is large enough, so $f_T''(x) = f_T(x)$ when $|x|$ is large. This implies $f_T(x) = C_\pm e^{\mp x}$ as $x \to \pm\infty$. In particular, $x^2 f_T \in L^2$. Therefore the function $f_T(x)$ lies in the domain of the operator H. We see that both functions $f(x)$ and $f_T(x)$ belong to domains of H and H_T.

Our goal is to show that $r_T(x) = f_T(x) - f(x) \to 0$ in $L^2(\mathbb{R})$ as $T \to \infty$. One has

$$H_T r_T(x) = (H - H_T)f(x) = w_T(x)f(x) \tag{1}$$

where

$$w_T(x) = \begin{cases} 0, & \text{when } |x| \leq T \\ x^2 - 1, & \text{when } |x| > T. \end{cases}$$

Typeset by $\mathcal{AMS}$-TEX

Let us multiply both sides of (1) by $r_T(x)$ and integrate the resulting equality over the real line:

$$(2)\qquad (H_T r_T, r_T) = \int_{|x|>T} (x^2-1) f(x) f_T(x) dx - \int_{|x|>T} (x^2-1)|f(x)|^2 dx.$$

Let

$$a_T = \int_{|x|>T} x^2 |f(x)|^2 dx \text{ and } b_T = \int_{|x|>T} x^2 |f_T(x)|^2 dx.$$

We use the Cauchy–Schwarz inequality to conclude from (2) that

$$(3)\qquad (H_T r_T, r_T) \leq a_T + \sqrt{a_T b_T}.$$

To finish the proof, it is sufficient to show that a_T go to 0 as $T \to \infty$ and b_T is uniformly bounded. Indeed $H_T \geq cI$ fo some $c > 0$; so $||r_T||^2 \leq c^{-1}(H_T r_T, r_T)$. Let us start from a_T. We multiply the equation $-f'' + x^2 f = g$ by $f(x)$ and integrate the resulting equality:

$$\int_{-\infty}^{\infty} |f'(x)|^2 dx + \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx = \int_{-\infty}^{\infty} f(x) g(x) dx.$$

Therefore,

$$\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \leq ||f|| \cdot ||g|| < \infty.$$

This implies $\lim_{T\infty} a_T = 0$.

We have already noticed that the function $f_T(x)$ is a decaying exponential function at both infinities. Therefore, the function $h_T(x) = x f_T(x)$ is a rapidly decaying function. It satisfies the equation

$$-h_T''(x) + V_T(x) h_T(x) + 2 f_T'(x) = x g(x).$$

We multiply both sides by $h_T(x)$ and integrate the resulting equation over the real line:

$$\int_{-\infty}^{\infty} |h_T'(x)|^2 dx + \int_{-\infty}^{\infty} V_T(x) |h_T(x)|^2 dx + 2 \int_{-\infty}^{\infty} x f_T(x) f_T'(x) dx = \int_{-\infty}^{\infty} x^2 g(x) f_T(x) dx.$$

We use integration by parts to get

$$2 \int_{-\infty}^{\infty} x f_T(x) f_T'(x) dx = \int_{-\infty}^{\infty} x (f_T^2(x))' dx = -||f_T||^2.$$

Therefore

$$b_T \leq \int_{-\infty}^{\infty} V_T(x) |h_T(x)|^2 dx \leq 2||f_T||^2 + ||x^2 g|| \cdot ||f_T||.$$

The right hand side of the last inequality is uniformly bounded because $||f_T|| \leq c^{-1}||g||$.