

## TAKE-HOME EXAM. DUE ON NOVEMBER 30

### PROBLEM 1

Let  $R > 0$ , and let  $B_R$  be the space of all  $C^\infty$  functions on  $\mathbb{R}$  such that

$$\|f\|_R = \sum_{k=0}^{\infty} \frac{R^k}{k!} \sup_x |f^{(k)}(x)| < \infty.$$

- a) Show that every function  $f \in B_R$  extends to a bounded, holomorphic function in the strip  $\{z \in \mathbb{C} : |\operatorname{Im} z| < R\}$ .
- b) Prove that  $B_R$  is a Banach algebra (multiplication is the usual multiplication of functions).
- c) Show that  $\sin x \in B_R$  for all  $R > 0$  and find the spectrum of  $\sin x$  in  $B_R$  (the answer will depend on  $R$ ).
- d) Let  $0 < R' < R$ . Show that  $d/dx$  is a bounded operator from  $B_R$  to  $B_{R'}$ .

### PROBLEM 2

Let  $E$  be a Banach space,  $B_n, B, A \in \mathcal{L}(E)$ . Suppose that the operator  $A$  has finite rank.

- a) Prove that if  $B_n \rightarrow B$  in the strong topology then  $B_n A \rightarrow BA$  in the norm topology.
- b) Does the convergence of  $B_n$  to  $B$  in the strong topology imply the convergence of  $AB_n$  to  $AB$  in the norm topology? Give a proof or a counterexample.

### PROBLEM 3

Let  $A$  be a bounded operator in a Banach space  $E$ . Prove that if for every  $x \in E$  there exists a non-trivial polynomial  $p_x(t)$  such that  $p_x(A)x = 0$  then there exists a non-trivial polynomial  $p(t)$  such that  $p(A) = 0$ .

### PROBLEM 4

Let  $L$  and  $R$  be the left shift and right shift operators in  $l^2(\mathbb{Z}_+)$ .

- a) Find the spectrum of  $L + R$ .
- b) Prove that 2 belongs to the spectrum of  $L + R^k$  for every integer  $k \geq 1$ .

### PROBLEM 5

Let  $K(x, y)$  be a continuous function on  $[0, 1]^2$ . Let  $A : L^2([0, 1]) \rightarrow L^2([0, 1])$  be an operator given by the formula

$$Au(x) = \int_0^x K(x, y)u(y)dy.$$

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a) Prove that there exists a constant  $C$  such that

$$\|A^k\| \leq \frac{C^k}{k!}$$

for all integer  $k \geq 0$ .

b) Find the spectrum of  $A$ .