RIESZ PROJECTIONS

Let T be a bounded operator acting in a Hilbert space H. The resolvent $R_T(\lambda) = (\lambda I - T)^{-1}$ is a holomorphic operator-valued function in an open set $\rho(T) = \mathbb{C}\setminus\sigma(T)$ of regular points of the operator T. Recall that $\{z \in \mathbb{C} : |z| > \|T\|\} \subset \rho(T)$. Let $\Omega \subset \mathbb{C}$ be a bounded domain with rectifiable boundary $\Gamma = \partial\Omega$. Assume that $\Gamma \subset \rho(T)$. We define a bounded operator

$$P_\Omega = \frac{1}{2\pi i}\int_\Gamma R_T(\lambda)d\lambda. \tag{1}$$

First, we prove that P_Ω is a projection.

Proposition 1. $P_\Omega^2 = P_\Omega$.

Proof. If $\epsilon > 0$ is a sufficiently small number then $\Omega_\epsilon = \{z \in \mathbb{C} : \operatorname{dist}(x, \Omega) < \epsilon\}$ is also a domain with rectifiable boundary and $\bar{\Omega}_\epsilon \setminus \Omega \subset \rho(T)$. Here I used the fact that $\rho(T)$ is an open set. Let Γ' be the boundary of Ω_ϵ. By Cauchy's Theorem,

$$P_\Omega = \frac{1}{2\pi i}\int_{\Gamma'} R_T(\lambda)d\lambda.$$

Then

$$P_\Omega^2 = \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} R_T(\mu)R_T(\lambda)d\mu d\lambda.$$

Hilbert's identity tells us that $R_T(\mu)R_T(\lambda) = (R_T(\lambda) - R_T(\mu))/(\mu - \lambda)$. Therefore,

$$\begin{aligned} P_\Omega^2 &= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} \frac{R_T(\lambda) - R_T(\mu)}{\mu - \lambda} d\mu d\lambda \\ &= \frac{1}{(2\pi i)^2}\int_\Gamma R(\lambda)d\lambda \int_{\Gamma'} \frac{d\mu}{\lambda - \mu} \\ &- \frac{1}{(2\pi i)^2}\int_{\Gamma'} R(\mu)d\mu \int_\Gamma \frac{d\lambda}{\lambda - \mu} = P_\Omega \end{aligned}$$

because

$$\int_{\Gamma'} \frac{d\mu}{\lambda - \mu} = 2\pi i \quad \text{and} \quad \int_\Gamma \frac{d\lambda}{\lambda - \mu} = 0.$$

Operator P_Ω is called *the Riesz projection* that correspond to the domain Ω. One can easily extend the definition of the Riesz projection to the case of unbounded domains with rectifiable boundary. Let $R > \|T\|$ be an arbitrary number, and let $\Omega_R = \Omega \cap \{z \in \mathbb{C} : |z| < R\}$. We define

$$P_\Omega = P_{\Omega_R}. \tag{2}$$

Cauchy's Theorem implies that the definition does not depend on the choice of $R > \|T\|$.

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Proposition 2. *Let* Ω *be a domain with rectifiable boundary* $\Gamma \subset \rho(T)$*, and* $\Omega^c = \mathbb{C} \setminus \bar{\Omega}$ *be the complement of the closure of* Ω*. Then* $P_\Omega + P_{\Omega^c} = I$*.*

Proof. Let $R > \|T\|$, and let $B_R = \{z \in \mathbb{C} : |z| < R\}$. For $\lambda \in C_r = \partial B_R$, the resolvent $R_T(\lambda)$ can be expanded into the Neumann series

$$R_T(\lambda)\frac{I}{\lambda} + \sum_{k=1}^{\infty} \frac{T^k}{\lambda^{k+1}}.$$

This series converges in norm uniformly on C_R, so it can be integrated term-by-term. One gets $P_{B_R} = I$. The statement of the Proposition 2 follows from this fact.

Let H_Ω be the range of P_Ω. It is a closed subspace in H. It follows from Proposition 2 that $H_\Omega \cap H_{\Omega^c} = \{0\}$ and $H_\Omega + H_{\Omega^c} = H$. We will combine these two facts in the following notation $H = H_\Omega \dot{+} H_{\Omega^c}$. Notice that $H = H_\Omega$ and H_{Ω^c} need not be orthogonal to each other, so I use $\dot{+}$ rather than $\oplus$.

The resolvent $R_T(\lambda)$ commutes with the operator T. Therefore, all Riesz projections associated with T commute with T. This implies that H_Ω and H_{Ω^c} are invariant subspaces of T. Indeed, let $x \in H_\Omega$. Then $P_\Omega T x = T P_\Omega x = Tx$, which means that $Tx \in H_\Omega$. Let T_Ω be the restriction of the operator T to H_Ω (H_Ω is a Hilbert space on its own right.)

Proposition 3.. *Let* Ω *be a domain with rectifiable boundary* $\Gamma \subset \rho(T)$*. Then* $\sigma(T_\Omega) \subset \Omega$*,* $\sigma(T_{\Omega^c}) \subset \Omega^c$*, and* $\sigma(T) = \sigma(T_\Omega) \cup \sigma(T_{\Omega^c})$*.*

Proof. The last statement follows from

$$\lambda I - T = P_\Omega(\lambda I - T)P_\Omega + P_{\Omega^c}(\lambda I - T)P_{\Omega^c},$$

so $\lambda I - T$ is invertible if and only if both $\lambda I - T_\Omega$ and $\lambda I - T_{\Omega^c}$ are invertible.

To prove the first statement, let us take $\mu \in \Omega^c$. In the case when the domain Ω is bounded, I define an operator

$$S(\mu) = \frac{1}{2\pi i} \int_\Gamma \frac{R_T(\lambda)}{\mu - \lambda} d\lambda.$$

If Ω is unbounded then I replace it by Ω_R, with $R > \|T\|$. The identity

$$(\mu I - T)R_T(\lambda) = I + (\mu - \lambda)R_T(\lambda)$$

implies

$$(\mu I - T)S(\mu) = S(\mu)(\mu I - T) = P_\Omega. \tag{3}$$

Notice that H_Ω is an invariant subspace for the operator $S(\mu)$. This is true because H_Ω is an invariant subspace for each operator $R_T(\lambda)$. Then, the identity (3) says that the restriction of $S(\mu)$ to H_Ω is the inverse to $\mu I - T_\Omega$. In particular, $\mu \in \rho(T_\Omega)$.