PROBLEM SET 1

Problem 1

Let $A : V_1 \to V_2$ and $B : V_2 \to V_3$ be linear mappings of finite index. Prove that the mapping $BA : V_1 \to V_3$ is of finite index, and

$$\operatorname{ind}(AB) = \operatorname{ind}(A) + \operatorname{ind}(B).$$

Problem 2

Let

$$0 \xrightarrow{A_0} V_1 \xrightarrow{A_!} V_2 \xrightarrow{A_2} \cdots \xrightarrow{A_{n-1}} V_n \xrightarrow{A_n} 0$$

be a sequence of linear mapping between finite dimensional linear spaces (the first one and the last one are zero-dimensional spaces) such that $R(A_j) \subset N(A_{j+1})$, $j = 0, \dots, n-1$. Prove that

$$\sum_{j=1}^{n} (-1)^j \dim(N(A_j)/R(A_{j-1})) = \sum_{j=1}^{n} (-1)^j \dim(V_j).$$

Problem 3

Let A be a symmetric $n \times n$ matrix with real entries. Let

$$\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)$$

be all eigenvalues of A; every eigenvalue is counted as many times as its multiplicity is.

a) Prove that

$$\lambda_n(A) = \max\{(Ax, x) : \ ||x|| = 1\}$$

where x is an n-tuple, $(x, y) = \sum x_j y_j$, and $||x||^2 = (x, x)$.

b) Prove that $\lambda_n(A)$ is a convex function on the space Symm_n of $n \times n$ symmetric matrices with real entries.

c) Prove that

$$\left\{ A \in \mathrm{Symm}_n : \ \sum_{j=1}^{n} \lambda_j(A) = 0 \right\}$$

is a linear subspace in Symm_n.

d) Prove that the set

$$\left\{ A \in \mathrm{Symm}_n : \ \sum_{j=1}^{n} \lambda_j(A)^2 \le 1 \right\}$$

is convex and find all its extreme points.