PROBLEM SET 3

PROBLEM 1

Let $2 \leq p < \infty$. Prove that there exists a constant $C > 0$ such that

$$\frac{|z|^p + |w|^p}{2} - \left|\frac{z+w}{2}\right|^p \geq C|z-w|^p$$

for any complex numbers z and w. Use this inequality to prove that the unit ball in an L^p space, $2 \leq p < \infty$, is uniformly convex.

PROBLEM 2

Let E be a Banach space. Suppose that the identity

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$$

holds for every $x, y \in E$. Prove that the formula

$$(x,y) = \frac{||x+y||^2 - ||x-y||^2}{4} + i\frac{||x+iy||^2 - ||x-iy||^2}{4}$$

defines a scalar product in E, and $||x||^2 = (x,x)$.

PROBLEM 3

Let P be a projection operator in a Hilbert space H. This means that $P \in \mathcal{B}(H)$ and $P^2 = P$. Suppose that $||P|| = 1$. Prove that P is an orthogonal projection.

PROBLEM 3

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk in the complex plane. We endow it with the two-dimensional Lebesgue measure $dxdy$; here $z = x + iy$. Show that the functions $f_k(z) = z^{k-1}$, $k = 1, 2, \ldots$, form an orthogonal system in $L^2(D)$. Let $\mathcal{L}$ be the closure of the span of the functions f_k. Find a formula for the orthogonal projection P onto $\mathcal{L}$:

$$Pf(\zeta) = \int_{\mathbb{D}} K(\zeta, z)f(z)dxdy.$$

Try to find as explicit expression for $K(\zeta, z)$ as possible.

Let A be a bounded operator in a Hilbert space H. The *numerical range* of A is the set

$$W(A) = \{(Ax, x) : ||x|| = 1\}.$$

1

PROBLEM 4

Let $a(x)$ be a (complex-valued) continuous function on the interval $[0, 1]$. Let M_a be the operator in $L^2([0,1])$ of multiplying by $a(x)$: $M_a f(x) = a(x) f(x)$. Prove that the closure of $W(M_a)$ is the convex hull of the range of the function $a(x)$. What can you say about the set $W(M_a)$? (*Hint:* you may find the notion of an extremal point of a convex set useful.) What will change if one takes $a(x) \in L^\infty([0,1])$, not necessarily continuous?

PROBLEM 5

Let S_r be the right shift operator in the space l^2 of sequences $x = \{x_n\}$, $n = 1, 2, \ldots$:

$$(S_r x)_n = \begin{cases} x_{n-1}, & \text{if } n > 1 \\ 0, & \text{if } n = 1. \end{cases}$$

Prove that

$$W(S_r) = \{\zeta \in \mathbb{C} : |\zeta| < 1\}.$$