PROBLEM SET 4

Problem 1

a) Let A be a bounded operator in a Hilbert space H. Let $x \in H$, $||x|| = 1$ and $||Ax|| = ||A||$. Prove that x is an eigenvector for A ($Ax = \lambda x$.)
b) Can the statement a) be extended to an arbitrary Banach space? Prove or give a counter-example.

Problem 2

Let α be an irrational number. Define an operator T_α on the space $L^2(S^1)$ of square integrable (over the period) 1-periodic functions by the formula
$T_\alpha(x) = u(x - \alpha)$. Prove that the sequence of operators

$$A_{\alpha,n} = \frac{1}{n} \sum_{k=0}^{n-1} T_\alpha^k$$

converges strongly and find its strong limit.
Hint. Look at how the operators $A_{\alpha,n}$ act on the functions $e_p(x) = e^{2\pi pix}$.

Problem 3

Let E_1 and E_2 be Banach spaces, and let A be a linear operator from E_1 to E_2; it is not assumed to be bounded. Suppose that $x_n \to x$ in E_1 implies $Ax_n \xrightarrow{w} Ax$ in E_2. Prove that A is bounded.

Problem 4

Consider $L^\infty(X, \mathcal{M}, \mu)$ as a subspace of the space of bounded operators in $L^2(X, \mathcal{M}, \mu)$ (μ is a sigma-finite measure): a function $\phi \in L^\infty$ is identified with the operator $M_\phi : L^2 \to L^2$ acting according to the formula $M_\phi f = \phi f$. Prove that the topology in L^∞ that is induced by the weak topology in $\mathcal{L}(L^2, L^2)$ is the same as the weak* topology in L^∞ as the dual space to L^1.

Problem 5

Let H be a separable Hilbert space. Let $\mathcal{R} \subset \mathcal{L}(H)$ be the set of all finite rank operators in H. Prove that the closure of $\mathcal{R}$ in the weak topology of $\mathcal{L}(H)$ is $\mathcal{L}(H)$.

Problem 6

Prove that a sequence $x_n \in l^1$ converges weakly in l^1 if and only if it converges in the norm topology.