PROBLEM SET 1

Problem 1

Let A and B be *strictly* positive unbounded operators in a Hilbert space H. Let $D(q_A)$ and $D(q_B)$ be domains of quadratic forms associated to A and B ($q_A(x) = (Ax, x)$, $q_B(x) = (Bx, x)$). We say that $A \geq B$ if $D(q_A) \subset D(q_B)$ and $q_A(x) \geq q_B(x)$ for every $x \in D(q_A)$. Notice that A^{-1} and B^{-1} are bounded non-negative self-adjoint operators. Prove that $A \geq B$ implies $A^{-1} \leq B^{-1}$ (this means $(A^{-1}x, x) \leq (B^{-1}x, x)$ for every $x \in H$.)

Hint. First, prove the following statement. Let $h \in H$, and $\Phi_{A,h}(x) = q_A(x) - 2\mathrm{Re}(x, h)$, $x \in D(q_A)$. Then the functional $\Phi_{A,h}$ is bounded from below on $D(q_A)$, and it attains its minimal value on $D(q_A)$. Find the point of minimum and the minimal value.

Problem 2

Consider symmetric operator $A_0 = (-d^2/dx^2) + \alpha/x^2$ in $L^2([0, \infty])$, with the domain $C_0^\infty((0, \infty))$. Here α is a positive number. Prove that the operator A_0 is closable. Find the domain of its closure and the domain of the corresponding quadratic form. Show that A_0 is essentially self-adjoint if $\alpha > 3/4$, and the defficiency indexes of A_0 equal 1 when $\alpha \leq 3/4$. In that case, describe all self-adjoint extensions of A_0.