PROBLEM SET 5

Problem 1

Let Ω be a bounded domain in $\mathbb{R}^n$, and let

$$d = \sup_{x,y\in\Omega} |x-y|$$

be the diameter of Ω. Let $K(z) \in L^1(B_d)$ where B_d is the ball of radius d in $\mathbb{R}^n$ centered at the origin. Define an operator

$$Au(x) = \int_\Omega K(x-y)u(y)dy.$$

a) Show that A is a bounded operator in $L^2(\Omega)$ and

$$||A|| \le \int_{B_d} |K(z)|dz.$$

(*Hint.* You may consider (Au, v) with $||u|| = ||v|| = 1$.)
b) Prove that A is a compact operator in $L^2(\Omega)$.

Problem 2

For a bounded domain Ω in the complex plane, we denote by $A(\Omega)$ the Banach space of functions that are analytic in Ω and continuous in $\bar{\Omega}$; the norm in $A(\Omega)$ is $||u|| = \max |u(x)|$. Let Ω and Ω_1 be bounded domains in $\mathbb{C}$ such that $\bar{\Omega}_1 \subset \Omega$.
a) Prove that the operator d/dz that maps a function to its derivative is a compact operator from $A(\Omega)$ to $A(\Omega_1)$.
b) Let

$$A = \sum_{k=0}^{n} a_k(z)\frac{d^k}{dz^k}$$

be a differential operator with coefficients $a_k(z) \in A(\Omega)$. Prove that A is a compact operator from $A(\Omega)$ to $A(\Omega_1)$.

Problem 3

Let A be a bounded operator in a Hilbert space H. Prove that A is compact if and only if the operator A^*A is compact.

Problem 4

Let a_j, l_j, and w_j, $j = 1, 2, \ldots$ be sequences of positive numbers such that $\{a_j\}$ decreases, $w_j < a_{j+1}$ and the series $\sum(a_j + l_j)$ converges. Let $b_1 = 0$ and

$$b_k = a_1 + 2a_2 + \cdots + 2a_{k-1} + a_k + 2l_1 + \cdots + 2l_{k-1}$$

for $k > 1$. Define closed sets in $\mathbb{R}^2$:

$$R_k = ([b_k - a_k, b_k + a_k] \times [-a_k, a_k]) \cup ([b_k + a_k, b_k + a_k + 2l_k] \times [-w_k, w_k]).$$

Let Ω be the interior of the union of all R_k. Let $\chi(t)$ be a smooth function on $[0, \infty)$ such that $\chi(t) = 1$ when $t < 1/2$ and $\chi(t) = 0$ when $t \geq 1$. Let $u_k(x, y)$ be a sequence of functions in Ω that is defined in the following way

$$u_k(x,y) = \begin{cases} 0, & \text{if } x < b_k - a_k - l_{k-1} \text{ or } x > b_k + a_k + l_k \\ \chi((b_k - a_k - x)/l_{k-1}), & \text{if } b_k - a_k - l_{k-1} \leq x \leq b_k - a_k \\ 1, & \text{if } b_k - a_k < x < b_k + a_k \\ \chi((x - b_k - a_k)/l_k), & \text{if } b_k + a_k \leq x \leq b_k + a_k + l_k. \end{cases}$$

a) Show that

$$\frac{\int_\Omega |\nabla u_k|^2 dx}{\int_\Omega |u_k|^2 dx} \leq \frac{C}{a_k^2}\left(\frac{w_k}{l_k} + \frac{w_{k-1}}{l_{k-1}}\right) \tag{1}$$

where C is a constant.

b) Prove that if the sequence that appears on the right in (1) is bounded (ex.: $a_k = l_k = 1/k^2$, $w_k = 1/5k^7$) then the embedding of $H^1(\Omega)$ to $L^2(\Omega)$ is not compact.