PROBLEM SET 6

Problem 1

Let T_1 and T_2 be compact operators in a Hilbert space. Prove that

$$s_{p+q-1}(T_1T_2) \leq s_p(T_1)s_q(T_2)$$

for every $p, q \geq 1$.

Problem 2

Let S be a positive, compact operator in a Hilbert space H. Let $\lambda_1 \geq \lambda_2 \geq \cdots$ be its eigenvalues; each eigenvalue is counted as many times as its multiplicity is. Prove the following formulas

$$\lambda_k = \sup_{L \subset H, \ \dim L = k} \inf_{x \in L} \frac{(Sx, x)}{||x||^2}$$

and

$$\lambda_k = \inf_{M \subset H, \ \mathrm{codim} M = k-1} \sup_{x \in M} \frac{(Sx, x)}{||x||^2}.$$

Problem 3

a) Let S_1 and S_2 be positive, compact operators, and let $\lambda_1(S_i) \geq \lambda_2(S_i) \geq \cdots$ be their eigenvalues, $i = 1, 2$. Assume that $S_2 - S_1$ is an operator of rank $r < \infty$. Prove that

$$\lambda_k(S_1) \geq \lambda_{k+r}(S_2)$$

for every k.
b) Let T be the Volterra operator in $L^2([0, 1])$,

$$Tu(x) = \int_0^x u(y)dy.$$

Prove that

$$s_k(T) = \frac{1}{\pi k} + O(k^{-2}).$$

Problem 4

Use the formula

$$\frac{d}{dt} \log \det(I + T(t)) = \mathrm{tr}(T'(t)(I + T(t))^{-1})$$

to prove that

$$\det((I + T_1)(I + T_2)) = \det(I + T_1) \det(I + T_2)$$

for trace class operators T_1 and T_2.