

## Symmetric Operator and their extensions

1. Let  $A$  be a densely defined linear operator in a Hilbert space  $H$ . We say that a complex number  $\lambda$  is a *point of regular type* for  $A$  if

$$\|(\lambda I - A)x\| \geq C\|x\|, \quad x \in D(A) \quad (1)$$

where  $C$  is a positive constant and  $D(A)$  is the domain of  $A$ .

**Notations.** By  $R_\lambda(A)$  I will denote the range of the operator  $\lambda I - A$ .

**Proposition 1.** *Let  $A$  be a closed operator, and let  $\lambda$  be a point of regular type. Then  $R_\lambda(A)$  is closed.*

**Proof.** Let  $y_n = \lambda x_n - Ax_n \rightarrow y$ . Then  $\|x_n - x_m\| \leq \|y_n - y_m\|/C$  (see (1).) Therefore,  $\{x_n\}$  is a Cauchy sequence, and  $x_n \rightarrow x$ . On the other hand,  $Ax_n \rightarrow \lambda x - y$ . The operator  $A$  is closed, so  $x \in D(A)$  and  $Ax = \lambda x - y$ . We conclude that  $y = (\lambda I - A)x \in R_\lambda(A)$ .

QED

**Exercise.** Show that if  $A$  is a closable operator and  $\lambda$  is a point of regular type for  $A$  then  $R_\lambda(\bar{A}) = \overline{R_\lambda(A)}$ .

**Definition.** Let  $A$  be a closable operator in a separable Hilbert space  $H$ , and let  $\lambda$  be a point of regular type for  $A$ . The codimension of the subspace  $R_\lambda(\bar{A})$  in  $H$  is called *or the deficiency number of  $\lambda$* , and it will be denoted by  $\text{def}(\lambda; A)$  or  $m(\lambda; A)$ .

**Remark.** For a closed operator  $A$ ,  $\lambda$  is a regular point iff it is a point of regular type and  $m(\lambda; A) = 0$ .

**Theorem 2.** *Let  $A$  be a closable operator in a separable Hilbert space  $H$ . The set of points of regular type of  $A$  is open, and  $m(\lambda; A)$  is constant on each connected components of this set.*

**Proof.** 1. Let  $\lambda$  be a point of regular type for  $A$ ; so (1) holds. Take any point  $\mu$  such that  $|\mu - \lambda| < C/2$ . Then

$$\|(\mu I - A)x\| \geq \|(\lambda I - A)x\| - |\lambda - \mu|\|x\| \geq (C/2)\|x\|, \quad (2)$$

so  $\mu$  is a point of regular type.

2. To prove that  $m(\lambda; A)$  is constant on connected components of the set of points of regular type, one has to show that the function  $m(\lambda; A)$  is continuous in  $\lambda$ . Let  $\lambda$  be a point of regular type such that (1) holds, and let  $|\mu - \lambda| < C/4$ . Then (2) holds. We will show that  $\dim R_\lambda(A)^\perp = \dim R_\mu(A)^\perp$ .

**Lemma 3.** *Let  $L_1$  and  $L_2$  be subspaces of a Hilbert space  $H$ , and let  $P_j$  be the orthogonal projection onto  $L_j$ ,  $j = 1, 2$ . If  $\|P_1 - P_2\| < 1$  then  $\dim L_1 = \dim L_2$ .*

**Proof.** The restriction of  $P_1$  to  $L_2$  is a one-to-one mapping into. In fact, if  $P_1x = 0$  and  $x \in L_2$ , then  $\|(P_1 - P_2)x\| = \|-x\| = \|x\|$ , and  $x = 0$ . Therefore,  $\dim L_2 \leq \dim L_1$ . Clearly, the opposite inequality also holds.

QED

By  $\Theta(L_1, L_2)$  we denote the number  $\sup_{x \in L_1 \setminus \{0\}} \|x - P_2x\|/\|x\|$ .

**Exercise.** Show that

$$\|P_1 - P_2\| = \max\{\Theta(L_1, L_2), \Theta(L_2, L_1)\}.$$

To finish the prove of the theorem, it is sufficient to show that both  $\Theta(R_\lambda(A), R_\mu(A))$  and  $\Theta(R_\mu(A), R_\lambda(A))$  are smaller than 1. Indeed, the difference between the orthogonal projections onto  $R_\lambda(A)^\perp$  and  $R_\mu(A)^\perp$  is the same as the difference between orthogonal projections onto  $R_\lambda(A)$  and  $R_\mu(A)$ . One has

$$\Theta(R_\lambda(A), R_\mu(A)) = \sup_x \inf_y \frac{\|(\lambda I - A)x - (\mu I - A)y\|}{\|(\lambda I - A)x\|} \leq \sup_x \frac{\|(\lambda - \mu)x\|}{\|(\lambda I - A)x\|} \leq \frac{1}{4}.$$

In the last inequality, to get an upper bound for the infimum over  $y$ , I took  $y = x$ . The number  $\Theta(R_\mu(A), R_\lambda(A))$  can be estimated in a similar way.

QED

**Exercise.** Let  $A$  be a closed operator. Show that  $R_\lambda(A)^\perp = \ker(\bar{\lambda}I - A^*)$ .

2. An operator  $A$  is called symmetric if  $(Ax, y) = (x, Ay)$  for every  $x, y \in D(A)$ . Equivalently,  $A \subset A^*$ . The last characterization implies that symmetric operators are always closable. Another useful characterization of the class of symmetric operators is given by the following proposition.

**Proposition 4.** *An operator  $A$  is symmetric iff the quadratic form  $(Ax, x)$ ,  $x \in D(A)$ , is real-valued.*

**Proof.** The "only if" part is obvious. Let us prove the "if" part. Take  $x, y \in D(A)$ . Then

$$0 = \Im(A(x + iy), x + iy) = \Im i((Ay, x) - (Ax, y)) = \Re((Ay, x) - (Ax, y)).$$

Therefore,

$$\Re(Ax, y) = \Re(Ay, x) = \Re(\overline{(Ay, x)}) = \Re(x, Ay)$$

for all  $x, y \in D(A)$ . Replacing  $x$  by  $ix$ , we conclude that  $\Im(Ax, y) = \Im(x, Ay)$ , and  $(Ax, y) = (x, Ay)$ .

QED

Let  $A$  be a symmetric operator. I will show now that all points outside the real axis are points of regular type for  $A$ . Let  $\lambda = \sigma + i\tau$ ,  $\tau \neq 0$ . Then

$$\|(\lambda I - A)x\| \geq |((\lambda I - A)x, x)|/||x|| \geq |\Im(\lambda I - A)x, x)|/||x|| = |\tau|||x||.$$

I have used the fact that  $(Ax, x)$  is a real number. Theorem 2 implies that  $m(\lambda; A)$  is constant in the upper half plane: we denote this constant by  $m_+(A)$ . It is also constant in the lower half plane: that constant is denoted by  $m_-(A)$ . If there is at least one real point of regular type then the whole set of points of regular type is connected, and  $m_+(A) = m_-(A)$ . Otherwise, these two numbers may be different.

From this point I will fix a complex number  $\lambda$  from the upper half plane. The next theorem tells us exactly what the domain of  $A^*$  is.

**Theorem 5.** *Let  $A$  be a closed symmetric operator, and let  $\lambda$  be a non-real complex number. Then*

$$D(A^*) = D(A) \dot{+} R_\lambda(A)^\perp \dot{+} R_{\bar{\lambda}}(A)^\perp.$$

**Proof.** Both  $R_\lambda^\perp$  and  $R_{\bar{\lambda}}^\perp$  are eigenspaces of  $A^*$  (the corresponding eigenvalues are  $\bar{\lambda}$  and  $\lambda$ ), so they lie inside of  $D(A^*)$ . This proves the  $\supset$  inclusion.

Let  $x \in D(A^*)$ . The vector  $\lambda x - A^*x$  can be decomposed as a sum of a vector from  $R_\lambda(A)$  and a vector from  $\ker(\bar{\lambda}I - A^*)$ ; the last vector will be denoted by  $(\lambda - \bar{\lambda})z$ :

$$\lambda x - A^*x = \lambda y - Ay + (\lambda - \bar{\lambda})z.$$

Here  $y \in D(A)$  and  $A^*z = \bar{\lambda}z$ . The last equality implies

$$A^*(x - y - z) = \lambda(x - y - z).$$

Therefore,  $w = x - y - z \in R_{\bar{\lambda}}(A)^\perp$ .

What remains to be proved is that  $x + y + z = 0$ ,  $x \in D(A)$ ,  $A^*y = \bar{\lambda}y$ ,  $A^*z = \lambda z$  implies  $x = y = z = 0$ . Indeed, we apply the operator  $A^*$  to the equation  $x + y + z = 0$  to get  $Ax + \bar{\lambda}y + \lambda z = 0$ . We subtract the last equality from  $\lambda x + \lambda y + \lambda z = 0$ :  $(\lambda I - A)x + (\lambda - \bar{\lambda})y = 0$ . The terms in the last equality belong two two mutually orthogonal subspaces of  $H$ . Therefore, both of them must vanish. In particular,  $y = 0$ . Similarly,  $z = 0$ .

QED

**Corollary.** A closed symmetric operator  $A$  is self-adjoint iff  $m_+(A) = m_-(A) = 0$ .

3. Let  $B$  be a symmetric extension of  $A$ . Then  $A \subset B \subset B^* \subset A^*$ . In particular this means that

$$D(B) \subset D(A^*) = D(A) + R_\lambda(A)^\perp + R_{\bar{\lambda}}(A)^\perp$$

and, on this domain,  $B$  acts like  $A^*$ . Keeping in mind Proposition 4, to find possible domains of symmetric extensions of  $A$ , we would like to know for which vectors  $x \in D(A^*)$  the imaginary part of  $(A^*x, x)$  vanishes. Let  $x = x_0 + x_1 + x_2$  where  $x_0 \in D(A)$ ,  $A^*x_1 = \bar{\lambda}x_1$ , and  $A^*x_2 = \lambda x_2$ . A simple computation shows that

$$\Im(A^*x, x) = \Im\lambda(\|x_2\|^2 - \|x_1\|^2).$$

Therefore,  $x$  may belong to  $D(B)$  only if

$$\|x_1\| = \|x_2\|. \quad (3)$$

On the other hand, if (3) holds for every  $x \in D(B)$  then the operator  $B$  is symmetric by Proposition 4. Let  $G = D(B) \cap (R_\lambda(A)^\perp + R_{\bar{\lambda}}(A)^\perp)$ . Then  $G$  is the graph of an isometry  $V$  from a subspace  $L \subset R_\lambda(A)^\perp$  into  $R_{\bar{\lambda}}(A)^\perp$ . We conclude that every symmetric extension of  $A$  can be obtained in the following way. Let  $V$  be an isometry from  $L \subset R_\lambda(A)^\perp$  into  $R_{\bar{\lambda}}(A)^\perp$ . Let  $\Gamma(V)$  be the graph of  $V$ . Then one takes  $D(B) = D(A) + \Gamma(V)$ , and  $B$  acts on its domain as  $A^*$ .

Let us compute  $m_\pm(B)$ . It is easy to see that  $R_\lambda(B) = R_\lambda(A) + L$  and  $R_{\bar{\lambda}}(B) = R_{\bar{\lambda}}(A) + VL$ . Therefore,  $m_+(B) = \dim R_\lambda(A)^\perp / L$  and  $m_-(B) = \dim R_{\bar{\lambda}}(A)^\perp / VL$ . In particular,  $m_\pm(B) = 0$ , and  $B$  is self-adjoint iff  $L = R_\lambda(A)^\perp$  and  $VL = R_{\bar{\lambda}}(A)^\perp$ . Such an isometry exists iff  $m_+(A) = m_-(A)$ . Let us formulate our conclusions as a theorem.

**Theorem 6.** *Let  $A$  be a closed symmetric operator in a separable Hilbert space  $H$ . It has a self-adjoint extension iff  $m_+(A) = m_-(A)$ . If this condition is met then every self-adjoint extension of  $A$  can be obtained by the following construction. Fix a non-real complex number  $\lambda$ , and let  $V$  be an isometry from  $R_\lambda(A)^\perp$  onto  $R_{\bar{\lambda}}(A)^\perp$ . Then  $D(A_V) = D(A) + \Gamma(V)$  where  $\Gamma(V)$  is the graph of  $V$ .*