

Weyl's asymptotic law

Let A be a self-adjoint operator in a Hilbert space, let $P(\mu) = P((-\infty, \mu))$ be the spectral projection associated to A , and let $N(\mu)$ be the dimension of the range of $P(\mu)$. I denote by $q = q_A$ the quadratic form $q(u, u) = (Au, u)$; the domain of q will be denoted by $\mathcal{D}(q)$.

Theorem 1. *Let*

$$\mathcal{L}_\mu = \{L \subset \mathcal{D}(q) : q(\phi, \phi) < \mu \|\phi\|^2; \quad \phi \in L, \phi \neq 0\}.$$

Then

$$N(\mu) = \sup_{L \in \mathcal{L}_\mu} \dim L.$$

Proof. 1. Let us show that

$$N(\mu) \leq \sup_{L \in \mathcal{L}_\mu} \dim L. \tag{1}$$

For $\lambda < \mu$, I define a subspace $L_{\lambda, \mu}$, the range of the spectral projection $P((\lambda, \mu))$. Clearly, $N(\mu) = \sup_{\lambda < \mu} \dim L_{\lambda, \mu}$. One has $L_{\lambda, \mu} \subset \mathcal{D}(A) \subset \mathcal{D}(q)$. If $\phi \in L_{\lambda, \mu}$ then

$$(A\phi, \phi) = \int_{(\lambda, \mu)} x(dP(x)\phi, \phi) < \mu \int_{(\lambda, \mu)} (dP(x)\phi, \phi) = \mu \|\phi\|^2.$$

Therefore $L_{\lambda, \mu} \in \mathcal{L}_\mu$, and (1) follows.

2. Let us prove

$$N(\mu) \geq \sup_{L \in \mathcal{L}_\mu} \dim L. \tag{2}$$

One can assume that $N(\mu) < \infty$ (otherwise, there is nothing to prove.) Take a subspace $L \subset \mathcal{D}(q)$ such that $\dim L > N(\mu)$. Then L has a non-trivial intersection with the image of the spectral projection $P([\mu, \infty))$. Take a non-zero vector ψ from this intersection. Then

$$(A\psi, \psi) = \int_\mu^\infty x(dP(x)\psi, \psi) \geq \mu \|\psi\|^2.$$

Therefore $L \notin \mathcal{L}_\mu$. This proves (2).

Q.E.D.

Proposition 2. *Let $N(\mu) < \infty$. Then, to the left of μ , the spectrum of A is discrete, and $N(\mu)$ equals to the number of eigenvalues of A that are smaller than μ , each eigenvalue counted as many times as its multiplicity is.*

Proof. The spectrum of A that is to the left of μ coincides with the spectrum of the restriction of A to the range of the spectral projection $P(\mu)$, which has dimension $N(\mu)$.

Q.E.D.

If $N(\mu) < \infty$ for all real numbers μ then the spectrum of A is discrete, and it concentrates at $+\infty$, so A is a semi-bounded operator with compact resolvent. Conversely, for semi-bounded operators with compact resolvent, $N(\mu)$ is always finite.

We will study the Laplacian

$$-\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \quad (3)$$

in a bounded domain $G \subset \mathbf{R}^n$. By $|G|$ I will denote the volume of G . Specifically, we will study two operators, both are given by the differential expression (3); the domain of the first one (the Dirichlet Laplacian) is $\{u(x) \in H^2(G) : u(x) = 0 \text{ on } \partial G\}$, and the domain of the second one (the Neumann Laplacian) is $\{u(x) \in H^2(G) : \partial u(x)/\partial \nu = 0 \text{ on } \partial G\}$ where ν is the outward normal direction to the boundary of G . To define the Neumann Laplacian in this way, one needs the boundary of G be at least differentiable almost everywhere. I will not worry about minimal possible assumptions on smoothness, and will assume that the boundary is differentiable (an extension to piecewise differentiable boundaries is straightforward). The quadratic form associated to both the Dirichlet and the Neumann Laplacians is given by the formula

$$q_{D,N}(u, u) = \int_G |\nabla u|^2 dx. \quad (4)$$

The quadratic form (4) is called *the Dirichlet functional*. The difference between these two problems is that the domain of q_D is $H_0^1(G) = \{u(x) \in H^1(G) : u(x) = 0 \text{ on } \partial G\}$, and the domain of q_N is $H^1(G)$.

Our first goal is to show that both the Dirichlet and the Neumann Laplacians are operators with compact resolvent.

Theorem 3. *Let G be a bounded domain in \mathbf{R}^n with differentiable boundary. Then $N_D(\mu) < \infty$ and $N_N(\mu) < \infty$ for all values of μ .*

Proof. First, we take the case of $G = Q_c^n$, a cube of side c . (The boundary of a cube is not quite smooth; it is only piecewise smooth, but this does not make any difference.) In a cube, both problems can be explicitly solved. The spectrum of the Dirichlet problem consists of the eigenvalues $\lambda_k = (\pi/c)^2 |k|^2$ where k is an integer vector, and $k_j > 0$. The spectrum of the Neumann problem consists of numbers given by the same formula; the difference is that here $k_j \geq 0$. Both functions $N_D(Q_c^n; \mu)$ and $N_N(Q_c^n; \mu)$ are finite, and a simple counting-of-lattice-points argument shows that

$$N_{D,N}(Q_c^n; \mu) = (2\pi)^{-n} v_n c^n \mu^{n/2} + O(\mu^{(n-1)/2}). \quad (5)$$

Here, v_n is the volume of the unit ball in \mathbf{R}^n .

Now, let G be a bounded domain. We place it inside of a cube Q_c^n of sufficiently large side c . Let $e_0 : H_0^1(G) \rightarrow H_0^1(Q_c^n)$ be the operator of extending a function by setting it equal to 0 outside of G . This operator preserves the functional (4). Therefore,

$$N_D(G; \mu) \leq N_D(Q_c^n; \mu) < \infty.$$

This finishes the proof of the theorem for the Dirichlet problem. The Neumann problem is more complicated because e_0 does not map $H^1(G)$ into $H^1(Q_c^n)$. To prove the Neumann-problem-statement I need a lemma.

Lemma. *Let G be a domain with differentiable boundary that lies inside a domain Ω . Then there exists a bounded extension operator $e : H^1(G) \rightarrow H^1(\Omega)$.*

To deduce the Theorem from the Lemma, notice that

$$N_N(G; \mu) \leq N_N(Q_c^n; \|e\|^2(\mu + 1))$$

where $\|e\|$ is the norm of an extension operator as an operator from $H^1(G)$ to $H^1(Q_c^n)$.
Q.E.D.

Now, we establish basic properties of $N_{D,N}(G; \mu)$. All the domains are assumed to be bounded, with differentiable boundary.

Property 1. $N_D(G; \mu) \leq N_N(G; \mu)$.

Indeed, the quadratic forms q_D and q_N are given by the same formula, and the domain of q_D is smaller than the domain of q_N .

Property 2. Let $G_1 \subset G_2$. Then $N_D(G_1; \mu) \leq N_D(G_2; \mu)$.

Indeed, $e_0 : H_0^1(G_1) \rightarrow H_0^1(G_2)$ is an isometric embedding.

Property 3. Let $\bar{G} = \bar{G}_1 \cup \bar{G}_2$ and $G_1 \cap G_2 = \emptyset$. Then

a. $N_D(G; \mu) \geq N_D(G_1; \mu) + N_D(G_2; \mu)$;

b. $N_N(G; \mu) \leq N_N(G_1; \mu) + N_N(G_2; \mu)$.

Property a. follows from existence of an isometric embedding

$$e : H_0^1(G_1) \oplus H_0^1(G_2) \rightarrow H_0^1(G).$$

One can define such an embedding by the formula

$$e(u_1, u_2)(x) = \begin{cases} u_1(x), & \text{if } x \in G_1; \\ u_2(x), & \text{if } x \in G_2. \end{cases}$$

Property b. follows from existence of an isometric embedding

$$r : H^1(G) \rightarrow H^1(G_1) \oplus H^1(G_2).$$

One can define such an embedding by the formula $r = (r_1, r_2)$, where r_j is the operator of taking the restriction of a function to G_j , $j = 1, 2$.

Now, I formulate and prove the Weyl asymptotic law for the Dirichlet problem.

Theorem 4. *Let G be a bounded domain in \mathbf{R}^n . Then*

$$N_D(\mu) = (2\pi)^{-n} v_n |G| \mu^{n/2} + o(\mu^{n/2}) \quad \text{as } \mu \rightarrow \infty \quad (6)$$

where v_n is the volume of the unit ball in \mathbf{R}^n .

Proof. Fix a positive number δ . For an integer point $k = (k_1, \dots, k_n) \in \mathbf{Z}^n$, I denote by $Q_{k,\delta}$ the cube $[\delta k_1, \delta(k_1 + 1)] \times \dots \times [\delta k_n, \delta(k_n + 1)]$. Let n_δ^- be the number of all such cubes that lie inside of G ; their union will be denoted by G_δ^- . Let n_δ^+ be the number of

the cubes that have non-empty intersection with G ; their union will be denoted by G_δ^\pm . Clearly,

$$n_\delta^\pm = \delta^{-n} |G_\delta^\pm|. \quad (7)$$

One deduces from properties 1–3 that

$$n_\delta^- N_D(Q_\delta^n; \mu) \leq N_D(G_\delta^-; \mu) \leq N_D(G; \mu) \quad (8)$$

and

$$N_D(G; \mu) \leq N_D(G_\delta^+; \mu) \leq N_N(G_\delta^+; \mu) \leq n_\delta^+ N_N(Q_\delta^n; \mu). \quad (9)$$

From (5), (7), and (8), one concludes

$$\liminf_{\mu \rightarrow \infty} \mu^{-n/2} N_D(G; \mu) \geq (2\pi)^{-n} v_n \delta^n n_\delta^- = (2\pi)^{-n} v_n |G_\delta^-|. \quad (10)$$

Similarly, formulas (5), (7), and (9) imply that

$$\limsup_{\mu \rightarrow \infty} \mu^{-n/2} N_D(G; \mu) \leq (2\pi)^{-n} v_n \delta^n n_\delta^+ = (2\pi)^{-n} v_n |G_\delta^+|. \quad (11)$$

Notice that $|G_\delta^\pm| \rightarrow |G|$ as $\delta \rightarrow 0$. The inequalities (10) and (11) hold for all values of δ , so

$$\liminf_{\mu \rightarrow \infty} \mu^{-n/2} N_D(G; \mu) \geq (2\pi)^{-n} v_n |G|$$

and

$$\limsup_{\mu \rightarrow \infty} \mu^{-n/2} N_D(G; \mu) \leq (2\pi)^{-n} v_n |G|.$$

The last two inequalities prove the theorem.

Q.E.D.

Problem. Let G be a bounded domain with differential boundary. Let $a_{ij}(x)$, $i, j = 1, \dots, n$, be real-valued differentiable functions in G that are continuous in \bar{G} . Assume that $a_{ij}(x) = a_{ji}(x)$ and the matrix $(a_{ij}(x))$ is positive definite for every $x \in \bar{G}$. Define quadratic forms q_D and q_N : both forms are given by the expression

$$q(u, u) = \int_G \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx;$$

the domain of q_D is $H_0^1(G)$, and the domain of q_N is $H^1(G)$.

1. Show that the forms q_D and q_N are closed.
2. What are self-adjoint operators L_D and L_N that correspond to these quadratic forms? Pay attention to boundary conditions!
3. Show that both operators L_D and L_N have compact resolvent.
4. Formulate and prove the asymptotics for the counting function $N_D(\mu)$ for the spectrum of the operator L_D .