PROBLEM SET 4

Problem 1

Let $U \subset \mathbb{R}^n$ be an open domain, and let $F : U \to \mathbb{R}^n$ be a C^1 mapping.
1) Let $x, y \in U$. Suppose that the whole segment $\{(1-t)x + ty : 0 \le t \le 1\}$ lies in U. Prove that

$$F(y) - F(x) = \int_0^1 F'((1-t)x + ty)(y-x)dt.$$

Here F' is the Jacobi matrix of F.
2) Let $x \in U$ and ϵ is small enough that the closed ball $\overline{B(x,\epsilon)}$ centered at x and of radius ϵ lies in U. Suppose that $\det F'(x) \ne 0$. For $y \in \overline{B(x,\epsilon)}$, define

$$R(y) = F'(x)^{-1}(F(y) - F(x)) - (y - x).$$

Prove that for every $\eta > 0$ there exists $\epsilon > 0$ such that

$$||R(y) - R(z)|| \le \eta ||y - z||$$

if $y, z \in \overline{B(x,\epsilon)}$.
3) Let $w \in \mathbb{R}^n$. Show that $F(y) = w$ if and only if $S_w(y) = y$ where

$$S_w(y) = x + F'(x)^{-1}(w - F(x)) - R(y).$$

4) Prove that there exist $\delta > 0$ and $\epsilon > 0$ such that S_w is a contraction in $\overline{B(x,\epsilon)}$ if $||w - F(x)|| \le \delta$.
5) Prove the Inverse Function Theorem:
Let U be a domain in $\mathbb{R}^n$, and let $F : U \to \mathbb{R}^n$ be a C^1 mapping. Let $x \in U$, and suppose that $\det F'(x) \ne 0$. Then there exist $\delta > 0$ and $\epsilon > 0$ such that for every $w \in B(F(x),\delta)$ there exists $y \in B(x,\epsilon)$ that solves the equation $F(y) = w$. This solution is unique, and the mapping

$$F^{-1} : B(F(x),\delta) \to B(x,\epsilon)$$

is differentiable. Moreover,

$$(F^{-1})'(w) = (F'(F^{-1}(w)))^{-1},$$

and, therefore, F^{-1} belongs to C^1.

From Spivak's book: problems 1, 3, 4, 13, 16, 17 p.p. 169–178

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