PROBLEM SET 3

Problem 1

Let $u(x,t)$ be a classical solution af the wave equation

$$\frac{\partial^2}{\partial t^2} = \Delta u$$

in $\mathbb{R}^3$. Let

$$U(t) = \sum_{|\alpha|\le 2} \int |D^\alpha u(x)| dx.$$

Here α is a four-dimensional multi-index; partial derivatives are taken in both spatial and time variables. Assume that $U(0) < \infty$.

a) Prove that there exists a constant C independent of u such that

$$|u(x,t)| \le \frac{C}{t} U(0) \text{ for } t > 0.$$

b) Prove that

$$\lim_{t\to\infty} \frac{U(t)}{t} = 0$$

implies $u(x,t) = 0$. *Hint.* Consider $v_T(x,t) = u(x, T-t)$ for large T.

Problem 2

Find a formula for the solution of the equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} - a^2 v$$

that satisfies initial conditions $v(x,0) = 0$, $v_t(x,0) = h(x)$; here a is a constant. *Hint.* The function

$$u(x_1, x_2, t) = v(x_1, t)\cos(a x_2)$$

satisfies the wave equation in $\mathbb{R}^2$.

Problem 3

Characterize all linear transformations such that

$$u(a_{11}x + a_{10}t, a_{01}x + a_{00}t)$$

is a solution of the wave equation whenever $u(x,t)$ solves the wave equation. Try to generalize your result to arbitrary dimensions.

Problems from Evans: 15, 17 on p.p. 88–89