TAKE-HOME EXAM
DUE ON DECEMBER 8

Problem 1

For what smooth functions $g(x)$ the problem

$$u_t + \frac{u_x}{1+u^2} = 0, \quad u(0,x) = g(x),$$

has a smooth solutions in the half-plane $\{(x,t) : t \geq 0\}$?

Problem 2

Prove that there exists a positive number ϵ such that the equation

$$\frac{\partial S}{\partial t} = \left(\frac{\partial S}{\partial x}\right)^2 + x^2 \tag{1}$$

with the initial condition

$$S(0,x) = \arctan x$$

has a smooth solution in the strip $\Pi_\epsilon = \{(x,t) : |t| < \epsilon\}$. Find the largest value of ϵ for which a smooth solution exists in the strip Π_ϵ.

Problem 3

Let $u(x)$ be a differentiable function defined in Ω_R, the exterior of the ball of radius R centered at the origin in $\mathbb{R}^3$. Suppose that $u(x)$ is continuous in the closure of Ω_R, that $u(x) = 1$ when x lies on the sphere of radius R (the boundary of Ω_R,) and that $u(x) \to 0$ when $x \to \infty$. Prove that

$$\int_{\Omega_R} |\nabla u(x)|^2 dx \geq 4\pi R,$$

and the equality takes place if and only if $u(x) = R/|x|$.

Problem 4

Let Ω be a bounded domain in $\mathbb{R}^n$, and let $g(x)$ be a continuous, non-negative function in Ω that vanishes on $\partial\Omega$, the boundary of Ω and that is not vanishing everywhere. Let $u(x,t)$ be the solution of the heat equation $u_t = \Delta u$ in Ω that satisfies the initial condition $u(x,0) = g(x)$ and the boundary condition $u(x,t) = 0$ when $x \in \partial\Omega$.
a) Prove that the function

$$a(t) = \int_\Omega u(x,t)dx$$

is strictly decreasing for $t \geq 0$.
b) Prove that there exists a constant C such that

$$\int_\Omega g(x)dx - a(t) \leq Ct \text{ for all } t > 0.$$

Problem 5

Let $u(x,t)$ be the solution of the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^3$, subject to the initial conditions $u(x,0) = 0$, $u_t(x,0) = h(x)$. Suppose that

$$\int |h(x)|^2 < \infty \text{ and } \int |\nabla h(x)|^2 < \infty.$$

For $R > 0$ we define

$$E_R(t) = \int_{|x|<R} \big[|\nabla u(x,t)|^2 + |u_t(x,t)|^2 \big] dx.$$

Prove that

$$\lim_{t\to\infty} E_R(t) = 0.$$

From Evans: Problems 3, 6, 7, p.163