Analytic continuation of the distribution $|x|^\lambda$

The distribution $|x|^\lambda \in \mathcal{D}'(\mathbf{R}^b)$ is defined for all complex numbers λ from the half-plane $\Re\lambda > -n$ by the formula

$$\langle |x|^\lambda, \phi\rangle = \int |x|^\lambda \phi(x) dx. \tag{1}$$

If a test function $\phi(x) \in \mathcal{D}(\mathbf{R}^n)$ is fixed then $\langle |x|^\lambda, \phi\rangle$ is a holomorphic function in λ in the half-plane $\Re\lambda > -n$. Our goal is to find its analytic continuation to as large domain as possible.

Choose a positive number N; the Taylor expansion of $\phi(x)$ reads

$$\phi(x) = \sum_{|\alpha|\le N} \frac{1}{\alpha!}\partial^\alpha\phi(0)x^\alpha + r_N(x)$$

where

$$|r_N(x)| \le C||\phi(x)||_{C^{N+1}}|x|^{N+1}. \tag{2}$$

We break the integral (1) into the sum

$$\int_{|x|\ge 1} |x|^\lambda\phi(x) + \int_{|x|\le 1} |x|^\lambda r_N(x) + \sum_{|\alpha|\le N}\frac{1}{\alpha!}\partial^\alpha\phi(0)\int_{|x|\le 1} x^\alpha|x|^\lambda dx. \tag{3}$$

The first term in (3) is defined for all values of λ; it is an entire in λ family of distributions. The second term in (3) is defined for $\Re\lambda > -n-N-1$, and it is holomorhic in λ in that half-plane; the estimate (2) implies that, for a fixed value of λ, it is a distribution. Let us take a look at the third term in (3). To compute the integral

$$\int_{|x|\le 1} x^\alpha|x|^\lambda dx$$

we use spherical co-ordinates $x = r\omega$ where ω belongs to the unit sphere S^{n-1}:

$$\int_{|x|\le 1} x^\alpha|x|^\lambda dx = \int_{S^{n-1}}\omega^\alpha d\omega\int_0^\infty r^{|\alpha|+\lambda+n-1}dr = \frac{c_\alpha}{\lambda+|\alpha|+n}.$$

So, the third term in (3) is defined for all values of λ except of $\lambda = -n-k$ where $k \in \mathbf{Z}_+$ is a non-negative integer number, as a functional of ϕ it is a combination of derivatives of the delta function, it is a meromorphic function of λ, and its residue at the pole $\lambda = -n-k$ equals

$$\sum_{\alpha|=k}\frac{c_\alpha}{\alpha!}\partial^\alpha\phi(0)$$

where

$$c_\alpha = \int_{S^{n-1}}\omega^\alpha d\omega. \tag{4}$$

Notice that if one of the α_j's is an odd number then the integrand in (4) is an odd function in ω_j, and $c_\alpha = 0$. Therefore $c_\alpha \neq 0$ only if $\alpha = 2\beta$ where $\beta \in \mathbf{Z}_+^n$. In particular, $|\alpha| = 2|\beta|$ is an even number, and the analytic continuation of $\langle |x|^\lambda, \phi \rangle$ is actually regular at points $\lambda = -n-k$ when k is an odd integer.

Because the number N that we chose in the beginning is arbitrary, we arrive to the following theorem.

Theorem. *The distribution $|x|^\lambda$, which is initially defined in the half plane $\Re\lambda > -n$, admits an analytic continuation to a meromorphic distribution-valued function in the whole complex plane. The only singularities of this continuation are simple poles at the points $\lambda_k = -n-2k$, $k = 0, 1, 2, \ldots$, and*

$$\mathrm{Res}_{\lambda=-n-2k}|x|^\lambda = \sum_{|\alpha|=2k} \frac{c_\alpha}{\alpha!} \partial^\alpha \delta(x) \tag{5}$$

where the constants c_α are given by (4).

We conclude these notes with computing explicitly the constants c_α. Consider the integral

$$I_\alpha = \int x^\alpha e^{-|x|^2} dx. \tag{6}$$

In spherical co-ordinates, it takes the form

$$I_\alpha = c_\alpha \int_0^\infty r^{n+|\alpha|-1} e^{-r^2} dr. \tag{7}$$

A substitution $r = \rho^2$ transforms the integral on the right in (7) into

$$\frac{1}{2}\int_0^\infty \rho^{\frac{|\alpha|+n}{2}-1} e^{-\rho} d\rho = \frac{1}{2}\Gamma\left(\frac{|\alpha|+n}{2}\right),$$

and therefore

$$c_\alpha = \frac{2I_\alpha}{\Gamma\left(\frac{|\alpha|+n}{2}\right)}. \tag{8}$$

To compute the integral I_α, consider a function

$$F(t_1, \ldots, t_n) = \int e^{-(t_1 x_1^2 + \cdots + t_n x_n^2)} dx = \pi^{n/2} t_1^{-1/2} \cdots t_n^{-1/2}.$$

We set $\alpha = 2\beta$ where $\beta \in \mathbf{Z}_+^n$ (otherwise, $I_\alpha = 0$.) Then

$$I_\alpha = (-1)^{|\beta|} \partial_t^\beta F(1, \ldots, 1). \tag{9}$$

Notice that

$$\frac{d^l}{dt^l}\left(t^{-1/2}\right) = (-1)^l 2^{-l} (2l-1)!!\, t^{-1/2-l}$$

where $(2l-1)!!$ is the product of all odd numbers from 1 to $2l-1$, and, in the case $l=0$, we set $(-1)!!=1$. Thus, (9) implies

$$I_\alpha = \pi^{n/2} 2^{-|\alpha|/2} (\alpha - 1)!!.$$

Finally,

$$c_\alpha = \frac{2^{-|\alpha|/2+1} \pi^{n/2} (\alpha-1)!!}{\Gamma\left(\frac{|\alpha|+n}{2}\right)} \tag{10}$$

when $\alpha = 2\beta$. One can substitute (10) into (5). Notice that

$$\frac{(2\beta)!}{(2\beta-1)!!} = 2^{|\beta|}\beta!;$$

so

$$\mathrm{Res}_{\lambda=-n-2k}|x|^\lambda = \sum_{|\beta|=k} \frac{\pi^{n/2} 2^{-2|\beta|+1}}{\beta!\Gamma\left(|\beta| + \frac{n}{2}\right)} \partial^{2\beta}\delta(x). \tag{5'}$$