The Stability of an Inverted Pendulum

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The stability of an inverted pendulum was analyzed by comparing experimental data to a theoretical model. First a simple model was derived, which treated the pendulum as a point mass. The experimental data was then collected using a trapezoidal prism attached to a jigsaw. Both the minimum frequency for stability and the region of stability were experimentally measured. The model was then revised and improved by accounting for the distributed mass of the physical pendulum.

Introduction

The equation of motion of the simple pendulum is one the most well-known problems in classical mechanics. There are myriad techniques to solve the simple oscillating motion of the pendulum as it lazily swings back and forth. If we consider the perfectly still pendulum hanging downwards to be at angle $\varphi = 0$, then we can explain that it exhibits this harmonic motion because it sits stably in a gravitational potential well centered around $\varphi = 0$. When we invert the pendulum, so that $\varphi = \pi$, it is at another equilibrium point. However, it is unstable in this position, and the slightest perturbation will cause it to topple back down. It then returns to harmonic motion about the stable equilibrium point $\varphi = 0$.

However, it is possible to cause this unstable equilibrium point to become stable. If small, rapid vertical oscillations are applied to the base, then the inverted pendulum becomes stable for a small range of angles near $\varphi = \pi$. The solution of this new equation of motion has immediate applications in many areas of science, most notably biomechanics where the head-neck-spine system may be examined as an inverted pendulum [1], [2]. Our research into this seemingly counterintuitive phenomenon compares our theoretical models with experimental results. We seek to understand the difference between the predicted and actual minimum frequency of stability and maximum stable deflection. Because we have two models, a simple point-mass model and a physical distributed mass model, we can also analyze the importance of taking the pendulum's shape into account.

Simple Model Theory

The derivation of the simple model can begin by looking through any textbook on classical dynamics [3]. In this model, the pendulum is treated as a point mass located at its center of mass. To find the equation of motion, the Lagrangian is used, which means that the kinetic and potential energy must be determined. If φ is defined as the angle of the pendulum from the downwards position, l as the distance between the base and the center of mass, m as the pendulum's total mass, and a and Ω as respectively the amplitude and frequency of the base's oscillations, then we can define the x and y coordinates of the point mass to be

$$x = l \sin \varphi$$

$$y = l \cos \varphi + a \cos \Omega t.$$

In order to calculate the kinetic energy, $T = \frac{1}{2} m v^2$, we must differentiate x and y. This gives us

$$v_x = l \dot{\varphi} \cos \varphi$$

$$v_y = -l \dot{\varphi} \sin \varphi - a \Omega \sin \Omega t.$$

Substituting in the velocities and using the property that $\sin^2 x + \cos^2 x = 1$, we have

$$T = \frac{m}{2} [l^2 \dot{\varphi}^2 + 2 l a \Omega \dot{\varphi} \sin \varphi \sin \Omega t + a^2 \Omega^2 \sin^2 \Omega t].$$

But the second term can be simplified using $\dot{\varphi} \sin \Omega t \sin \varphi = \Omega \cos \Omega t \cos \varphi - (\sin \Omega t \cos \varphi)'$. Because the quantity we are aiming for is actually $\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$, we can ignore any terms which are a complete derivative or a function of only t. Therefore, we are left with

$$T = \frac{m l^2}{2} \dot{\varphi}^2 + m l a \Omega^2 \cos \Omega t \cos \varphi.$$

The potential energy is very simple. Because we are on the surface of Earth, the gravitational potential energy is $U = -mgy$, where m is mass and g is Earth's gravitational acceleration.. This gives us

$$U = -mg(l\cos\varphi + a\cos\Omega t).$$

Or, because we can ignore the second term since it is a function of only t,

$$U = -mgl\cos\varphi.$$

The Lagrangian $\mathscr{L} = T - U$ is

$$\mathscr{L} = \frac{ml^2}{2}\dot{\varphi}^2 + mla\Omega^2\cos\Omega t\cos\varphi + mgl\cos\varphi,$$

so the equation of motion of the simple inverted pendulum is

$$\ddot{\varphi} + \left(\frac{g}{l} + \frac{a\Omega^2}{l}\cos\Omega t\right)\sin\varphi = 0.$$

It is not immediately obvious that this implies that the $\varphi = \pi$ position becomes stable with appropriate parameters. However, by separating φ into two oscillations, one which is large and slow (the swinging of the rod) and the other which is small and fast (the oscillation of the base), we can average and determine the stability condition. It becomes clear that this multiscale interaction does indeed cause the upwards position to become stable. We begin with

$$\varphi = \Phi(t) + \Psi(t),$$

where $\Phi(t)$ has a large amplitude but slow frequency, and $\Psi(t)$ has a small amplitude but fast frequency. This means that $\ddot{\varphi}$ may be written as

$$\ddot{\varphi} = -\frac{\partial u}{\partial\varphi} + f,$$

where $u = -\frac{g}{l}\cos\varphi$ and $f = -\frac{a\Omega^2}{l}\cos\Omega t\sin\varphi$.

Therefore,

$$\ddot{\varphi} = \ddot{\Phi} + \ddot{\Psi} \cong -\frac{\mathrm{du}}{\mathrm{d}\Phi} - \Psi\frac{\mathrm{d}^2\mathrm{u}}{\mathrm{d}\Phi^2} + \mathrm{f}(\Phi,\mathrm{t}) + \Psi\frac{\mathrm{df}}{\mathrm{d}\Phi}.$$

We know that Ψ is small, but because it has a fast frequency, $\ddot{\Psi}$ is large. Additionally, although Φ is large, it has a slow frequency. Balancing the large terms, we have

$$\ddot{\Psi} = \mathrm{f}(\Phi,\mathrm{t}),$$

and because Φ can be treated as a constant,

$$\Psi = -\frac{f}{\Omega^2}.$$

We also have

$$\ddot{\Phi} = -\frac{\mathrm{du}}{\mathrm{d}\Phi} - \Psi\frac{d^2u}{d\Phi^2} + \Psi\frac{df}{d\Phi}.$$

Averaging this with respect to the fast oscillations, we end up with

$$\ddot{\Phi} = -\frac{\mathrm{du}}{\mathrm{d}\Phi} + \langle\Psi\frac{df}{d\Phi}\rangle = -\frac{\mathrm{du}}{\mathrm{d}\Phi} - \frac{1}{\Omega^2}\langle\mathrm{f}\frac{df}{d\mathrm{t}}\rangle,$$

which can also be expressed as

$$\ddot{\Phi} = -\frac{du_{eff}}{d\Phi}$$

with an effective potential of

$$u_{eff} = u + \frac{1}{2\Omega^2}\langle f^2\rangle.$$

We then calculate $\langle f^2\rangle$:

$$\langle f^2\rangle = \frac{a^2\Omega^4}{l^2}\frac{1}{T}\sin^2\varphi\int_t^{t+T}\cos^2\Omega\tau\,d\tau = \frac{a^2\Omega^2}{4l^2}\sin^2\varphi.$$

And finally, we have an equation for the effective potential of the simple inverted pendulum:

$$u_{eff} = \frac{g}{l}\left(-\cos\varphi + \frac{a^2\Omega^2}{4gl}\sin^2\varphi\right).$$

$$\frac{du_{eff}}{d\varphi} = \frac{g}{l}\sin\varphi\left(1 + \frac{a^2\Omega^2}{2gl}\cos\varphi\right).$$

From this pair of equations, we can see that both the $\varphi = 0$ and $\varphi = \pi$ positions are stable. The stability condition for $\varphi = \pi$ is

$$\frac{a\Omega}{\sqrt{2gl}} > 1.$$

Assuming this stability condition is met, the potential graph has two maxima and two minima in the region $0 \leq \varphi < 2\pi$. The minima are the stable equilibrium points, and the maxima (located between 0 and π and π and 2π) represent the borders of stability. These maxima, which denote the range of stability about $\varphi = \pi$, can be determined by setting the derivative of the effective potential equal to zero.

Experimental Data

The pendulum we used to gather data consisted of a trapezoidal prism, which was only free to oscillate left and right, attached at the base to a jigsaw which provided the up-and-down oscillations. Before we began, we measured the dimensions of the rod. It had a length of 28 cm, the square base was 0.9 cm by 0.9 cm, and the rectangular end was 0.9 cm by 0.4 cm. The length between the base and the center of mass was both measured and calculated to be

$$l = 0.12\text{m}.$$

Next, we measured that the amplitude of small oscillations was

$$a = 0.009\text{m} \pm 0.0005\text{m}.$$

A stroboscope was used to measure the frequency of fast oscillations. The stroboscope lamp emits brief and rapid flashes of light, and the frequency of these flashes can be adjusted until it equals the frequency of the jigsaw. The stroboscope was in units of inverse minutes, which we converted to rad/s, and could be fine tuned to a precision of approximately ± 3 min^{-1}.

The minimum stable frequency was measured by starting the jigsaw at a low speed, and slowly increasing its frequency until the $\varphi = \pi$ position became stable. The minimum stable frequency was measured as

$$206 \pm 5.1 \text{ rad/s.}$$

The range of stability was determined by measuring the maximum stable deflection of the pendulum from $\varphi = \pi$ and using simple trigonometry. The range of stability measurements were taken at

$$220.0 \text{ rad/s.}$$

The range of stability was

$$0.553 \pm 0.0088 \text{ rad, or } 31.7° \pm 0.5° \text{ from } \varphi = \pi.$$

It was also necessary to measure the period of slow oscillations for the physical model explained later. We did this by using a stopwatch to time how long it took for various numbers of cycles to be completed. The jigsaw was not used for this portion. The period of slow oscillations was measured as

$$T = 0.821\text{s} \ \pm 0.006\text{s}$$

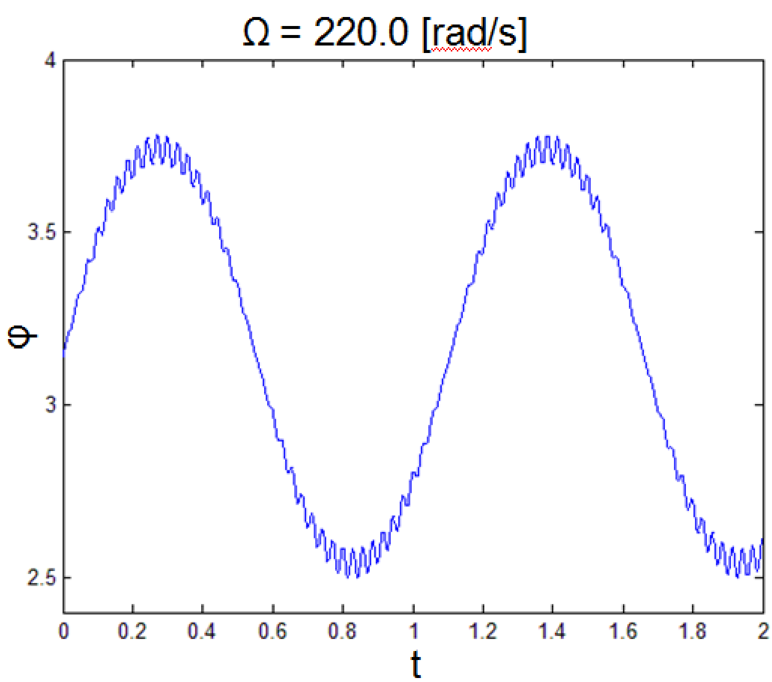
This gives us a slow oscillation frequency of

$$\omega_0 = 7.65 \text{ rad/s} \pm 0.06 \text{ rad/s}.$$

There are several probable sources of error. The stroboscope was difficult to fine tune, so there could be a slight error in the frequency measurements. In addition, the pendulum was loosely attached to the base, so there was slight wobbling in a third dimension. However, the largest source of error is likely due to the tape which held the jigsaw trigger (which controls frequency) constant. It had a tendency to loosen over time due to the vibrations. This could have significantly affected our stability range measurements.

Simple Model Results

The following graph of the angle vs. time demonstrates that, counterintuitively, the pendulum does oscillate about $\varphi = \pi$.



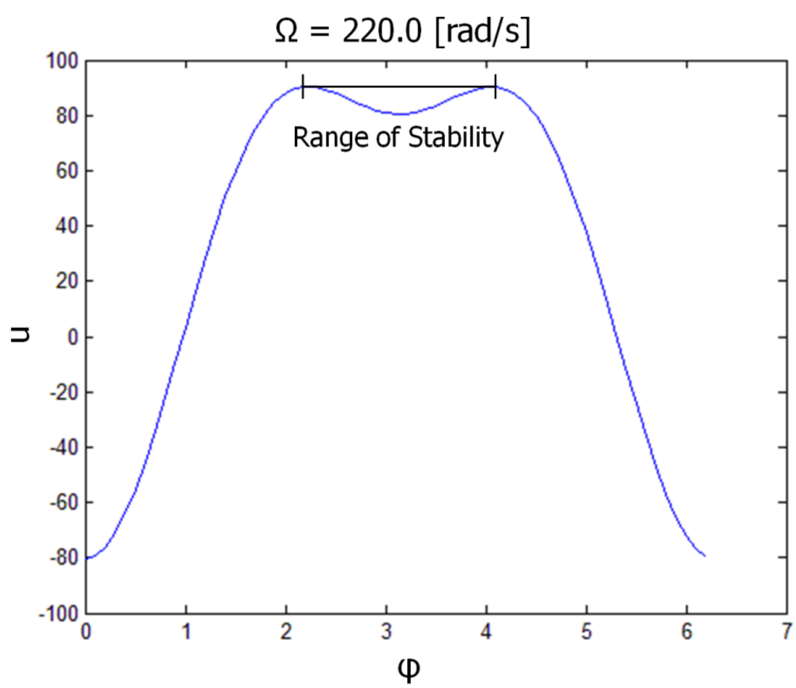
From the theory section, we know that stability occurs when $\frac{a\Omega}{\sqrt{2gl}} > 1$. Therefore, the minimum frequency of stability is

$$\Omega > \frac{\sqrt{2gl}}{a}.$$

Using the parameters for the jigsaw pendulum, we have

$$\Omega > 164.62 \text{ rad/s.}$$

It is easy to see the range of stability by graphing u_{eff} vs. angle. Clearly, there is a potential well centered on $\varphi = \pi$.



By taking the derivative of this potential and setting it equal to zero, we find that the maxima which delineate the potential well are located at $\varphi = 2.226$ and 4.057. Therefore, the range of stability is 0.915 radians, or $52.4°$ from π.

For this simple model, the pendulum was treated as though it were a point mass located at its center of mass. While this is a fairly good approximation for an object such as a sphere or disk at the end of a thin, rigid rod, these results show that it is important to take into account the mass distribution for a rod shaped pendulum like ours.

Physical Model

The derivation for the physical model resembles the simple model, with the key difference that the pendulum is treated as a distributed mass rather than a point mass. We begin again with the Lagrangian. This time, the x and y coordinates represent the position of each point of the pendulum, not merely of a single point mass:

$$x = l \sin \varphi$$

$$y = l \cos \varphi + a \cos \Omega t.$$

Differentiating x and y with respect to time, we obtain

$$\dot{x} = l\dot{\varphi} \cos \varphi$$

$$\dot{y} = -l\dot{\varphi} \sin \varphi - a\Omega \sin \Omega t.$$

The kinetic energy is $T = T_x + T_y$, and we must integrate over the pendulum's length to determine each component. The kinetic energy in x is

$$T_x = \frac{1}{2}\int \dot{x}^2 \, dm = \frac{1}{2}\dot{\varphi}^2 \cos^2 \varphi \int \rho(l) l^2 \, dl,$$

because $dm = \rho(l)\, dl$, while the kinetic energy in y is

$$T_y = \frac{1}{2}\int \dot{y}^2 \, dm$$

$$T_y = \frac{1}{2}\dot{\varphi}^2 \sin^2 \varphi \int \rho(l) l^2 \, dl + a\Omega\dot{\varphi} \sin \Omega t \sin \varphi \int \rho(l) l \, dl + \frac{1}{2} a^2 \Omega^2 \sin^2 \Omega t \int \rho(l) \, dl.$$

Combining these two kinetic energy terms and using $\sin^2 x + \cos^2 x = 1$ gives us

$$T = \frac{1}{2}\dot{\varphi}^2 \int \rho(l) l^2 \, dl + a\Omega\dot{\varphi} \sin \Omega t \sin \varphi \int \rho(l) l \, dl + \frac{1}{2} a^2 \Omega^2 \sin^2 \Omega t \int \rho(l) \, dl.$$

Because $\dot{\varphi} \sin \Omega t \sin \varphi = \Omega \cos \Omega t \cos \varphi - (\sin \Omega t \cos \varphi)'$ and since we can ignore any terms which are a complete derivative or a function of only t, we have

$$T = \frac{1}{2}\dot{\varphi}^2 \int \rho(l) l^2 \, dl + a\Omega^2 \cos \Omega t \cos \varphi \int \rho(l) l \, dl.$$

The gravitational potential energy is

$$U = -g \int \rho(l) l \, dl \cos \varphi + a\Omega \sin \Omega t,$$

or, ignoring the second term because it is a function of only t,

$$U = -g \int \rho(l) l \, dl \cos \varphi.$$

Define the moments $J = \int \rho(l) l^2 \, dl$ and $I = \int \rho(l) l \, dl$. The Lagrangian is then

$$\mathscr{L} = \frac{1}{2} J \dot{\varphi}^2 + a\Omega^2 I \cos \Omega t \cos \varphi + gI \cos \varphi,$$

so the equation of motion for the physical inverted pendulum is

$$\ddot{\varphi} + \left(\frac{gI}{J} + \frac{a\Omega^2 I}{J} \cos \Omega t \right) \sin \varphi = 0.$$

This is very similar to the previously derived equation for the simple model, with the important distinction that, instead of being divided by l, the second and third terms are multiplied by the ratio of moments I/J.

We will again separate φ into a large, slow oscillation and a small, fast oscillation, then average to determine an effective potential from which we can analyze the stability. We have

$$\varphi = \Phi(t) + \Psi(t),$$

where $\Phi(t)$ has a large amplitude but slow frequency, and $\Psi(t)$ has a small amplitude but fast frequency. $\ddot{\varphi}$ may again be written as

$$\ddot{\varphi} = -\frac{\partial u}{\partial \varphi} + f,$$

but this time $u = -\frac{gI}{J}\cos\varphi$ and $f = -\frac{a\Omega^2 I}{J}\cos\Omega t \sin\varphi$. Again,

$$\ddot{\varphi} = \ddot{\Phi} + \ddot{\Psi} \cong -\frac{\mathrm{du}}{\mathrm{d}\Phi} - \Psi\frac{\mathrm{d}^2\mathrm{u}}{\mathrm{d}\Phi^2} + \mathrm{f}(\Phi,\mathrm{t}) + \Psi\frac{\mathrm{df}}{\mathrm{d}\Phi}.$$

While Ψ is small, it has a high frequency, so $\ddot{\Psi}$ is large. Φ is large but has a slow frequency. Balancing the large terms, we again have

$$\ddot{\Psi} = \mathrm{f}(\Phi,\mathrm{t}),$$

and because Φ can be treated as a constant,

$$\Psi = -\frac{f}{\Omega^2} = \frac{a\mathrm{I}}{\mathrm{J}}\cos\Omega\mathrm{t}\sin\varphi.$$

Note that this is the same as before, but with a factor correcting for the distributed mass replacing l. Averaging this with respect to the fast oscillations, we also have

$$\ddot{\Phi} = -\frac{\mathrm{du}}{\mathrm{d}\Phi} - \frac{1}{\Omega^2}\langle \mathrm{f}\frac{df}{d\mathrm{t}}\rangle,$$

which can also be expressed as

$$\ddot{\Phi} = -\frac{du_{eff}}{d\Phi}$$

with an effective potential of

$$u_{eff} = u + \frac{1}{\Omega^2}\langle f^2\rangle.$$

We then calculate $\langle f^2\rangle$:

$$\langle f^2\rangle = \frac{a^2\Omega^4 I^2}{J^2}\frac{1}{T}\sin^2\varphi\int_t^{t+T}\cos^2\Omega\tau\,d\tau = \frac{a^2\Omega^4 I^2}{2J^2}\sin^2\varphi.$$

And, the equation for the effective potential of the physical inverted pendulum:

$$u_{eff} = \frac{gI}{J}\left(-\cos\varphi + \frac{a^2\Omega^2 I}{2gJ}\sin^2\varphi\right).$$

But $\omega_0^2 = \frac{gI}{J}$, where ω_0 is the frequency of the slow oscillation. Therefore,

$$u_{eff} = \omega_0^2\left(-\cos\varphi + \frac{a^2\Omega^2\omega_0^2}{2g^2}\sin^2\varphi\right)$$

$$\frac{du_{eff}}{d\varphi} = \omega_0^2\left(\sin\varphi + \frac{a^2\Omega^2\omega_0^2}{g^2}\sin\varphi\cos\varphi\right).$$

This gives us the stability condition for $\varphi = \pi$:

$$\frac{a\Omega\omega_0}{\sqrt{2}g} > 1.$$

The equation for the maximum angle of stability, found by setting the derivative of the potential with respect to the angle equal to zero, is

$$\varphi_{max} = \arccos\left(-\frac{g^2}{a^2\Omega^2\omega_0^2}\right).$$

Physical Model Results

Rearranging the above stability condition, we find that the minimum frequency of stability is

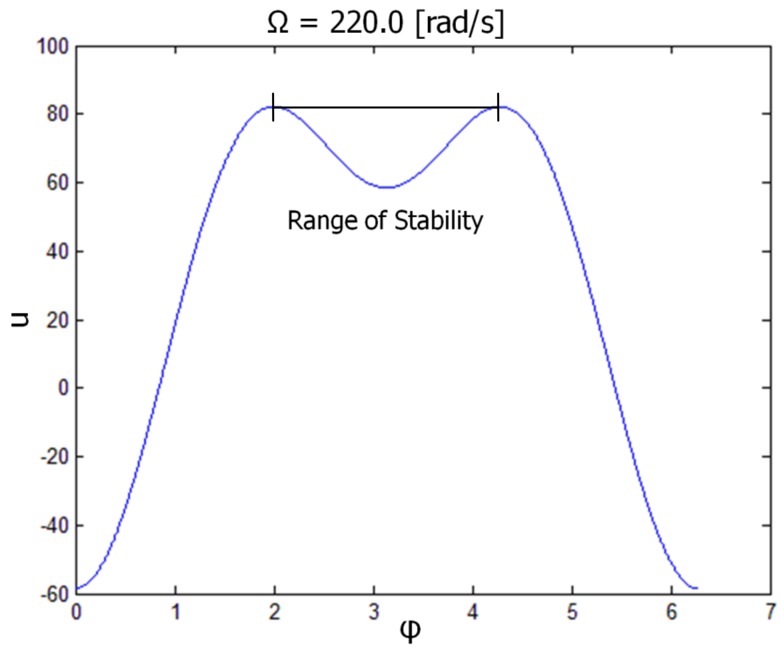
$$\Omega > \frac{\sqrt{2}g}{a\omega_0}.$$

Using the experimental parameters, we obtain

$$\Omega > 201.3\ \text{rad/s}.$$

This is much closer to experiment than the simple model's results.

The effective potential graph looks very similar to the previous, simple model one.



The range of stability is 1.14 rad, or 65.3° from π.

Conclusions

We have demonstrated that the upwards position of the pendulum becomes stable due to the multiscale interactions of large, slow oscillations and small, fast oscillations. With an appropriate amplitude and frequency of small oscillations, the unstable equilibrium point at $\varphi = \pi$ becomes a potential well.

Summarizing our results, we have

	Minimum Frequency of Stability [rad/s]	Range of Stability [rad]
Experimental Data	206 ± 5.1	0.553 ± 0.0088
Simple Model	164.62	0.915
Physical Model	201.3	1.14

While the physical model has a much more accurate minimum frequency of stability, it seems that neither model can correctly predict the range of stability. It is possible that some experimental error has thrown off the results, or simply that more work needs to be done to improve the model. In any case, it is clearly important to take into account the shape of the pendulum. Though a simple point mass model can approximate the minimum frequency of stability to a certain extent, the physical model does a much better job without even significantly increasing the difficulty of calculation.

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