Physical mechanisms of the Rogue Wave phenomenon

Final Report

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"We were in a storm and the tanker was running before the sea. This amazing wave came from the aft and broke over the deck. I didn't see it until it was alongside the vessel but it was special, much bigger than the others. It took us by surprise. I never saw one again." Philippe Lijour, first mate of the oil tanker Esso Languedoc, describing the huge wave that slammed into the ship off the east coast of South Africa in 1980. [5]

1 Summary

In this project, the rogue wave phenomenon is introduced along with its importance. The main equations governing both linear and nonlinear theory are presented. The three main linear theories proposed to explain the rogue rave phenomenon are presented and a linear model reproducing rogue waves due to dispersion is shown. A nonlinear model for rogue waves in deep and shallow water is also exhibit.

2 Introduction

Seafarers speak of "walls of water", or of "holes in the sea", or of several successive high waves ("three sisters"), which appear without warning in otherwise benign conditions. But since 70s of the last century, oceanographers have started to believe them. [4]

Storm wave height can reach up to 8-10 m in the deep sea under extreme wind conditions. Nevertheless, observations were indeed reported for suddenly emerged huge waves on an otherwise quiet and calm background wave field in deep water. Such waves are called rogue waves (or freak waves, monster waves, giant waves, steep waves, etc.). These waves can easily reach a wave height over 10 m without any warning and thus pose great dangers to ships.[6]

Naval architects have always worked on the assumption that their vessels are extremely unlikely to encounter a rogue. Almost everything on the sea is sailing under the false assumption that rogue waves are, at worst, vanishingly rare events. The new research suggest that's wrong, and has cost lives. Between 1969 and 1994 twenty-two super carriers were lost or severely damaged due to the occurrence of sudden rogue waves; a total of 542 lives were lost as a result. [5]

Freak, rogue or giant waves correspond to large-amplitude waves surprisingly appearing on the sea surface. Such waves can be accompanied by deep troughs (holes), which occur before and/or after the largest crest. [4]

There are several definitions for such surprisingly huge waves, but the one that is more popular now is the amplitude criterion of freak waves, which define them as waves such height exceeds at least twice the significant wave height:

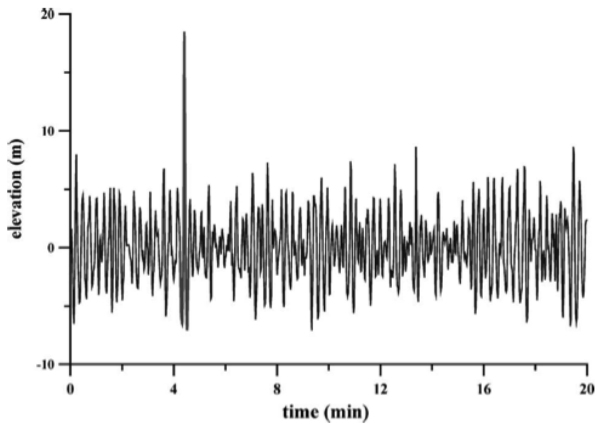

Figure 1: New year wave in the northern sea[4]

$$AI = \frac{H_{fr}}{H_s} > 2 \tag{1}$$

Where AI=abnormality index, H_{fr}=height of the freak wave, and H_s= significant wave height, which is the average wave height among one third of the highest waves in a time series (usually of length 10–30 min). In that way, the abnormality index (AI) is the only parameter defining whether the wave is rogue or not. [1]

3 Rogue Waves observations

According to orthodox oceanography, rogue waves are so rare that no ship or oil platform should ever expect to encounter one. But as the shipping lanes fill with supercarriers and the oil and gas industry explores ever-deeper parts of the ocean, rogue waves are being reported far more often than they should. The most spectacular sighting of recent years is probably the so-called New Year Wave, which hit Statoil's Draupner gas platforms in the North Sea on New Year's Day 1995. The significant wave height at the time was around 12 metres. But in the middle of the afternoon the platform was struck by something much bigger. According to measurements made with a laser, it was 26 metres from trough to crest.[5]

Hundreds of waves satisfying condition (1) have been recorded by now, and several waves with an abnormality index larger than three ($Ai > 3$) are known. [1] As an example, Figure 1 shows the New Year Wave, with an $AI = 3.19$.

4 Water waves equations

Force acting over most of the real fluid is composed of three contributions, namely pressure force, body force, and viscous force. Among these, only viscous forces have shear forces that change the rotational status of fluid particles. It is this shear forces that change the rotational status of fluid particles. Therefore, when viscous forces are neglected, the vorticity will be neither created nor destruyed. [6]

In the coastal region where the water depth is from a few meters to a few tens of meters, the boundary layer region is much smaller than the entire flow region. Therefore, it is justified to assume that the water waves can be governed by the Laplace equation based on the potential flow theory. [6]

Outside the boundary layer, the viscous effect diminishes toward the far field. This implies that in a location away from the solid body, the fluid gradually loses the driving mechanism that changes its vorticity status. If the fluid flow is initially irrotational, it will remain so. This type of flow is called irrotational flow.[6]

If the flow is irrotational, there exists a scalar velocity potential function ϕ that can be expressed as follows:

$$u = -\nabla\phi = -\left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right) \tag{2}$$

We can consider that water waves have been generated from a fluid that was initially at rest -that is, from an irrotational motion and the irrotationality condition implies that the flow satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi = \triangle\phi = 0 \tag{3}$$

Where $\triangle$ is the Laplacian operator $\triangle = \nabla \cdot \nabla$, (x, y, z)=cartesian coordinates, ϕ =velocity potential of the flow.

To solve (3), conditions on boundaries are needed. The fluid domain that is considered is bounded by two kinds of boundaries: the interface, which separates the air from the water; and the wetted surface of an impenetrable solid (the sea bottom, for instance). [1]

The kinematic boundary condition states that the normal velocity of the surface is equal to the normal velocity of the fluid at the surface, this condition can be represented with the following equation

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} - \frac{\partial\phi}{\partial z} = 0, \quad z = \eta \tag{4}$$

Where $\eta(x, y, t)$ represents the free surface elevation.

Since η and ϕ are both unknown on the free surface, a second boundary condition is needed: the dynamic boundary condition. This condition is derived from the Bernoulli equation and can be expressed as follows

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + gz = 0, \quad z = \eta \tag{5}$$

Where g=gravity acceleration.

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{P}{\rho} + gz = 0$$

Considering the sea bottom equation, $z = -h(x, y)$, (4) takes the form of the sea bottom boundary condition

$$\frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial h}{\partial y} + \frac{\partial \phi}{\partial z} = 0, \quad z = h(x, y) \tag{6}$$

Where h is the water depth.

The water wave problem reduces to solve the system of equations consisting of the Laplace equation (3), kinematic boundary condition (4), dynamic boundary condition (5) and sea bottom boundary condition (6).[1]

Although the Laplace equation (3) is a linear partial differential equation, the difficulty in solving water wave problems arises from the nonlinearity of kinematic and dynamic boundary conditions. [1]

5 Linear theory of water waves

The water wave equations, which are nonlinear, can be transformed into a sequence of linear problems by using a perturbation procedure. Let us assume the following perturbation expansions in the parameter ϵ for the unknowns ϕ and η: [3]

$$\phi(x, y, z, t) = \sum_{n=1}^{\infty} \epsilon^n \phi_n(x, y, z, t) \tag{7}$$

$$\eta(x, y, t) = \sum_{n=1}^{\infty} \epsilon^n \eta_n(x, y, t) \tag{8}$$

Where $\epsilon = ak$ is a linearization parameter that physically represents the wave steepness (an important measure in deep water) with a=wave amplitude and k=wavenumber (spatial frequency of the wave in radians per unit distance).

For small amplitude waves ($\epsilon \ll 1$), we can ignore the terms of order $O(\epsilon^n)$ with $n > 1$ in the previous expansions. Hence, the velocity potential and free surface elevation are approximated as

$$\phi(x, y, z, t) = \epsilon\phi_1, \tag{9}$$

$$\eta(x, y, t) = \epsilon\eta_1 \tag{10}$$

The corresponding linear system of equations to be solved is

$$\triangle\phi = 0, \quad -h < z < 0 \tag{11}$$

$$\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = 0 \tag{12}$$

$$\frac{\partial\phi}{\partial t} + g\eta = 0 \quad \text{on} \quad z = 0 \tag{13}$$

$$\frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial h}{\partial y} + \frac{\partial\phi}{\partial z} = 0, \quad z = -h \tag{14}$$

The following analytical solution of (11) satisfies all the constraints imposed from (12) to (14):

$$\phi = -\frac{H}{2}\frac{g \cdot cosh(k(h+z))}{cosh(kh)} sin(kx - \sigma t)$$

Where σ=wave angular frequency, $H = 2a$=wave height.

The validity of this solution relies on the assumptions of both $ka \ll 1$ and $a/h \ll 1$ (an important measure in shallow water) so that all the nonlinear terms can be neglected in the derivation.

From (13), the corresponding free surface displacement η is found to be: [6]

$$\eta = \frac{1}{g}\frac{\partial\phi}{\partial t} = \frac{H}{2} cos(kx - \sigma t) \tag{15}$$

This represents a progressive wave, in which the wave form moves from the left to the right direction without change of wave shape.

With the use of (12), the angular wave frequency σ is related to the wavenumber k, and the local water depth h, by the wave dispersion equation:

$$\sigma^2 = gk \cdot tanh(kh)$$

The above equation can be rewritten as:

$$c^2 = \frac{g}{k} tanh(kh) \tag{16}$$

Where c=phase velocity.

Equation (16) implies that waves with different wavenumbers (or wavelengths) will propagate at different speeds and therefore separate from each other. [6]Because of this, waves with similar frequency will group together and separate grom other wave groups. This process of self-sorting is called wave dispersion.

6 Linear mechanisms causing rogue waves

In linear theory, the wind wave field can be sought as the sum of a very large number of small-amplitude independent monochromatic waves with different frequencies and directions of propagations.[4]

Linear waves can be superposed together to create new waves. A random sea is regarded as the result of wave superposition of an infinite number of small linear waves with different wave amplitudes and frequencies traveling in different directions. [6]

Three main methods suggested in linear theory as the origin of rogue waves are mentioned, the first two are presented in a qualitatively manner, while the third one is discussed in more detail.

6.1 Geometrical or spatial focusing

In areas where waves from storms in the open sea approach shallower waters (e.g. several locations along the Norwegian coast), the waves will be refracted and diffracted. There may be focusing of wave energy in certain areas such that the probability of encountering large waves is much greater than in other areas. [2]

Amplification of water waves due to the effect of geometrical focusing is a well known process for waves of any physical nature. Directional sea-wave distribution occurs when the waves come from different directions: the open sea, several stormed areas, and for the coastal zone, due to the reflection from complicated coastal lines. The refracted waves may interfere, providing wave energy concentration near capes.[4]

A variable wind generates complex and variable structures of rays in storm areas, playing the role of initial conditions for the system. Caustics are very sensitive to the small variation of the initial conditions, and as a result, the caustics and focuses appear and disappear at “random” points and “random” times, providing rare and short-lived character of the freak wave phenomenon.

6.2 Wave-current interaction

The first theoretical models of the rogue wave phenomenon considered wave-current interaction because rogue waves were observed very often in the Agulhas current in the South African coast, where the Agulhas current is countered by westerlies.

When waves from one current are driven into an opposing current, this results in shortening of wavelength, causing and increase in wave height, and oncoming wave trains to compress together into a rogue wave.

Lavrenov (1998) calculated the ray pattern in the vicinity of the Agulhas Current for one event of freak wave ocurrence and showed that it contains focus points where the wave energy concentrates. [4]

6.3 Focusing due to dispersion

Due to strong dispersion of water waves, each individual sine wave travels with a frequency-dependent velocity, and may travel along different directions. At one moment, short waves with small group velocities are located in front of long waves with large group velocities, but then after some time the long waves will overtake the shorter waves and a large-amplitude wave can occur due to the spatio-temporal superposition. [1]

If during the initial moment the short waves with small group velocities are located in front of the long waves having large group velocities, then in the phase of development, long waves will overtake short waves, and large-amplitude wave can appear at some fixed time owing to the superposition of all the waves located at the same place. Afterward, the long waves will be in front of the short waves, and the amplitude of wave train will decrease. [4]

Generalizations of the kinematic approach in the linear theory can be done by using various expressions of the Fourier integral for the wave field near the caustics. In a generalized form, it has been expressed through the Maslov integral representation, described in detail for water waves by Brown (2001) and others. Brown pointed out the relationship between the focusing of a unidirectional wave field and “canonical” caustics: fold and longitudinal cusp. The simplified form of such a representation for the conditions of optimal focusing is presented

$$\eta(x,t) = \int_{-\infty}^{+\infty} \eta(\Omega) exp\left[i(\Omega T - kx)\right] d\Omega \tag{17}$$

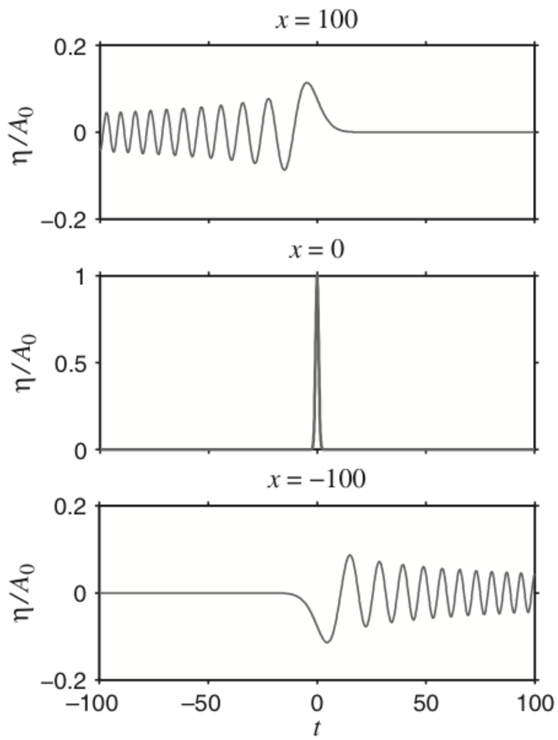

Figure 2: Formation of the freak wave of Gaussian form in shallow water[1]

Integral (17) can be calculated for smooth "freak waves" (initial data), for instance for a Gaussian pulse with amplitude, A_0 and width, k^{-1}, with the following equation for shallow water conditions[4]

$$\eta(x,t) = \frac{A_0}{k\sqrt[3]{\frac{h^2ct}{2}}} exp\left(\frac{1}{2h^2ctk^2}\left(x - ct + \frac{6}{77h^2ctk^4}\right)\right) \mathrm{Ai}\left(\frac{x - ct + \frac{9}{77h^2ctk^4}}{\sqrt[3]{\frac{h^2ct}{2}}}\right) \tag{18}$$

where A_0=Initial wave train amplitude.

This wave packet evolves into a Gaussian pulse, and then again disperses according to (18).

Equation (18) is presented in Figure 2, to show a formation of a rogue wave of Gaussian shape. Dimensionless variables are defined as $T = \Omega t - x\Omega/c$, $X = xh^2\Omega^3/(2c)$.

Figure 2 was reproduced in Matlab by the student and is showed in Figure 3, the parameters used are: $A_0 = 1m$ (initial amplitude), $h = 30m$ (water depth), $T = 15s$ (period), $x = 100m$ (distance to focusing point), $c = \sqrt{9.81h}$ (phase velocity), $\lambda = cT$ (wavelength), $k = 2\pi/\lambda$ (wavenumber), $\Omega = ck$ (frequency).

Figure 2 shows how from an initial pulse A_0, a rogue wave can be formed in a focusing point ($x = 0$) due to the superposition of small amplitude waves.

This effect is utilized in large wave tanks for testing of ship models. With a wave maker at the end of the tank one creates a signal in the form of a wave train where the wave length varies, with the shortest waves

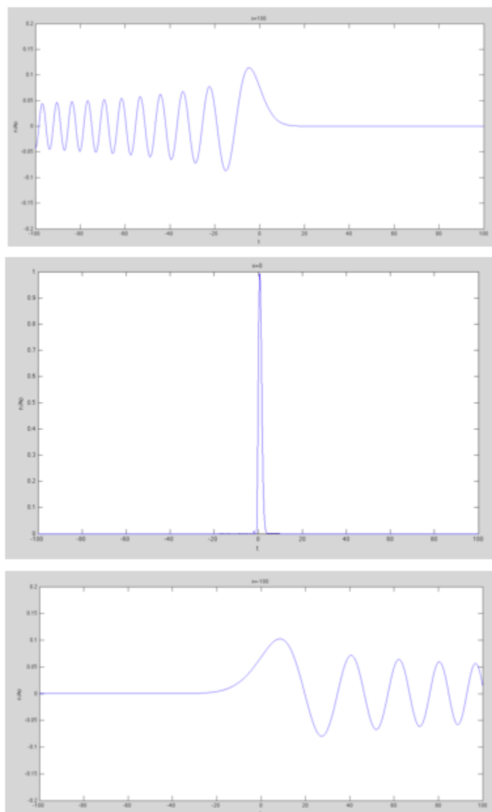

Figure 3: Reproduction of Figure 2 in Matlab

in front. Long waves propagate faster and will catch up on the shorter waves. Thereby a few large waves are created over a short time and within a limited area. [2]

7 Nonlinear theory of water waves

Linear wave theory is constructed on the assumption of $ka \ll 1$ and $a/h \ll 1$. When wave amplitude increase beyond a certain range, the linear wave theory may become inadequate. The reason is that those higuer order terms that have been neglected in the derivation become increasingly important as wave amplitude increases. [6]

The immediate extension of the linear wave theory is to retain all the second-order terms in (7) and (8) during the derivation. This will end up with the so-called second order Stokes wave theory. The theory stipulates that for larger amplitude waves, the wave profile is the sum of two sinusoidal waves, one of which is the same that obtained from the linear wave theory (15) and the other from the second-order correction terms, i.e: [6]

$$\eta = \frac{H}{2} cos(kx - \sigma t) + \frac{H^2 k}{16} \frac{cosh(kh)}{sinh^3(kh)} (2 + cosh(2kh)) cos(2(kx - \sigma t))$$

Note that for second-order Stokes waves, the linear dispersion equation (16) remains valid.

8 Nonlinear mechanisms causing rogue waves

8.1 Nonlinear wave interaction

Nonlinear wave interaction is the process during which waves exchange energy among different wave modes. Depending on the local wave and wind condition, wave energy can be transferred from high frequency to low frequency and viceversa.

Nonlinear wave interaction also results in the so-called Benjamin-Feir instability, which was named after the theoretical work by Benjamin and Feir in 1967. In their works, the difficulty in mantaining a permanent shape of nonlinear wave trains for long-distance propagation was explained theoretically. It was found that under certain circumstances, the wave mode of the fundamental frequency can become unstable and grow exponentially in time under the small disturbance of the wave modes with a close frequency.

Due to nonlinear effects, the crest at the peak of the envelope will propagate faster than the nearby waves, thus shortening the waves ahead and lenghtening the waves behind. The group velocity, at which the wave energy is transmitted, will increase behind but decrease in front with the change of wavelength. As a result, the crest at the peak of the envelope will continously gain net energy from the wave groups and develop instability. [6]

8.2 The Nonlinear Schrödinger equation

The generation of extreme wave events can be simply obtained from the Benjamin-Feir instability (or modulational instability) of uniformly traveling trains of Stokes waves in water of infinite and finite depths. Stokes' wave trains are unstable in terms of various perturbations. Among these instabilities is the Benjamin-Feir instability (a long-wave instability). Using a Hamiltonian approach, Zakharov discovered in 1968 the existence of modulational instability of Stokes waves. Furthermore, in the context of modulated water waves, he obtained the famous Nonlinear Schrödinger equation (NLS). [1]

The 1D NLS equation is written as follows

$$i\left(\frac{\partial A}{\partial t}+c_{gr}\frac{\partial A}{\partial x}\right)=\frac{\omega}{8k^2}\frac{\partial^2 A}{\partial x^2}+\frac{\omega k^2}{2}|A|^2A \tag{19}$$

where the surface elevation, $\eta(x,t)$ is given by

$$\eta(x,t)=\frac{1}{2}\left(A(x,t)exp(i(k_0x-\omega_0t))+c.c.+\cdots\right)$$

k=wavenumber, ω=frequency of the carrier wave, *c.c.*=complex conjugate, $(\cdots)$=highest armonics of the carrier wave. The complex wave amplitude, A, is a slowly varying function of x and t.

The simplest nonlinear Schrödinger equation has many exact solutions. One of them has been particularly popular as candidate to explain freak waves. It is called a "breather" and starts out as a periodic wave train where the amplitude is weakly modulated. After some time it develops a particularly strong focusing of wave energy by which a small part of the wave train "breathes" itself up at the expense of the neighborhood.[2]

One of the breather solutions (a singular breather on an infinite domain) corresponds to the so-called algebraic breather (in the system of coordinates moving with the group velocity) [4]

$$A(x,t)=A_0\cdot exp(i\omega t)\left[1-\frac{4(1+2i\omega t)}{1+16k^2x^2+4\omega^2t^2}\right] \tag{20}$$

This algebraic breather (in dimensionless coordinates, A/A_0,$A_0k_0^2x$, and $A_0k_0^2\omega_0t$) is shown in Figure 4. The maximal height of this wave (from trough to crest) exceeds 3. This model shows how the nonlinear instability of a weakly modulated wave train in deep water may generate rogue waves.

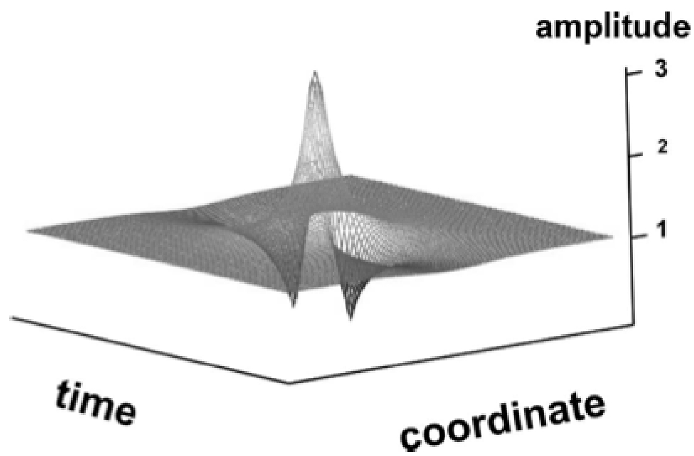

Figure 4: Algebraic breather as a model of abnormal wave in a time periodic wave train [4]

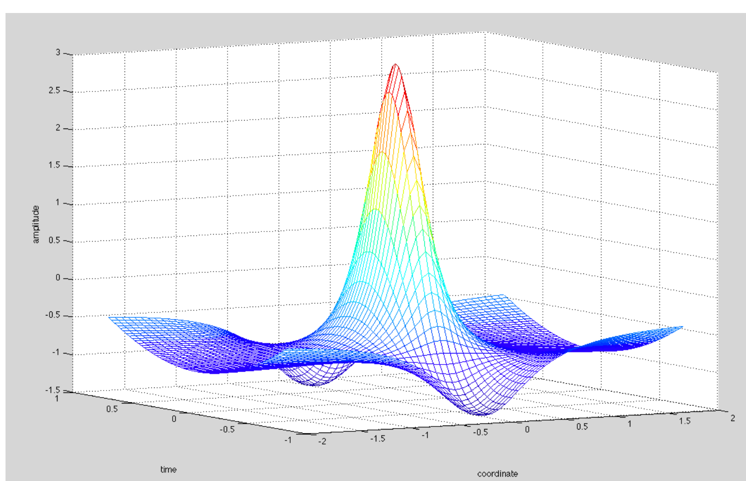

Figure 5: Reproduction of Figure 4 in Matlab

8.3 Shallow-Water Rogue Waves

When the sea becomes shallow, the water flow induced by surface waves is almost uniform with depth. Thus, properties of shallow water waves are radically different from those in deep water: the wave dispersion is weak, and waves are now affected by the seafloor. [1]

For shallow water the ratio of nonlinearity to dispersion is usually high and the generation of the highest harmonics become more effective. The simplified model of 2D unidirectional waves in the shallow water taking into account weak nonlinearity and dispersion is the Korteweg-de Vries equation [4]

$$\frac{\partial \eta}{\partial t} + c\left(1 + \frac{3\eta}{2h}\right)\frac{\partial \eta}{\partial x} + \frac{ch^2}{6}\frac{\partial^3 \eta}{\partial x^3} = 0 \tag{21}$$

where $c = \sqrt{gh}$ is the long wave speed.

The Korteweg-de Vries (KdV) equation (21), is a basic weakly dispersive and weakly nonlinear model. This equation was the first that exhibited exact soliton solutions. The soliton solution is a steady-state solution of (21) [1]:

$$\eta(X,T) = Hsech^2\left(\sqrt{\frac{3H}{4D}}\frac{X - VT}{D}\right), \quad V = C\left[1 + \frac{H}{2D}\right] \tag{22}$$

where D=water depth, H=wave height, V=velocity of the soliton.

The solutions of (21) are stable, and, therefore, the nonlinear mechanism of the rogue wave formation due tu modulation instability does not apply in shallow water. If the initial wave field presents weakly modulated wave train, its form is modified in the process of the wave evolution, buth the wave amplitude does not vary significantly. Thus, the rogue wave can appear only due to focusing mechanism. [4]

The Korteweg-de Vries equation can be transformed with the following dimensional variables,

$$\zeta = \frac{\eta}{h}, \quad y = \frac{x - ct}{h}, \quad \tau = \frac{ct}{h} \tag{23}$$

and (21) becomes [7]

$$\frac{\partial \zeta}{\partial \tau} + \frac{3}{2}\zeta\frac{\partial \zeta}{\partial y} + \frac{1}{6}\frac{\partial^3 \zeta}{\partial y^3} = 0 \tag{24}$$

Then, (24) is solved in periodic domain on interval $L = 1600$ by using a finite-difference scheme. At the first stage, an initial condition representing the expected freak wave is chosen as the Gaussian impulse: [7]

$$\zeta(y,0) = A_0 exp\left[-\left(\frac{x - 5L/6}{d}\right)^2\right] \tag{25}$$

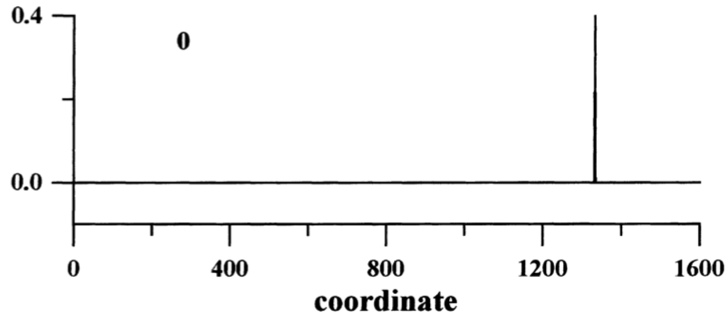

Figure 6: Evolution of an impulse disturbance into solition for t=0 in the framework of the Korteweg-de Vries equation. [7]

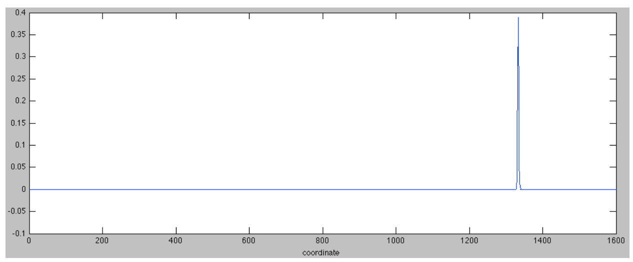

Figure 7: Matlab reproduction of Figure 6

The amplitude A_0 and characteristic width d are varied in numerical experiments. A graphic of (25) is presented in Figure 6. This plot was reproduced in Matlab and is shown in Figure 7. The values used are $A_0 = 0.4$ and $d = 2$. Figure 7 shows how a rogue wave is formed approximately at $x = 1270$.

9 Conclusions

Several observations of rogue waves have been reported in recent years. Because of the danger that rogues represent to ships, significant progress has been made in their study. Nevertheless, the precise physical mechanisms causing them remain unknown. Linear and nonlinear theories have been developed to explain rogues. The linear theories are a consequence of solving the Laplace equation, linearizing both the kinematic and dynamic boundary conditions. These linearization can be done for small amplitude waves. In the linear theory, rogue waves can be seen as a supersposition of small amplitude waves focused on a point. When wave amplitude increases, the linear theory can't be applied and a nonlinear approach is needed. The nonlinear theories are obtained by solving the Laplace equation but keeping nonlinear the kinematic and dynamic boundary conditions. In the linear theory, rogue waves can be explained due to the instability of nonlinear waves. Theories developed to explain rogues shown that they can occur in both deep and shallow water.

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