

Notes on Inverted Pendulum: Straightforward Approach

A. Equation of a shaking pendulum: Let (x, y) be a fixed coordinate frame. The end of a pendulum moving in the y -direction periodically. The movement is with a magnitude A and a frequency Ω . Let l be the length of the pendulum and ϕ be the angle in between the pendulum and the y -axis. For the pendulum we have

$$y = l \cos \phi + A \cos \Omega t, \quad x = l \sin \phi.$$

From which we obtain

$$\dot{y} = -l \sin \phi \dot{\phi} - A \Omega \sin \Omega t$$

$$\ddot{y} = -l \cos \phi (\dot{\phi})^2 - l \sin \phi \ddot{\phi} - A \Omega^2 \cos \Omega t$$

$$\dot{x} = l \cos \phi \dot{\phi}$$

$$\ddot{x} = -l \sin \phi (\dot{\phi})^2 + l \cos \phi \ddot{\phi}.$$

Use the second law of mechanics $\mathbf{F} = m\mathbf{a}$ and decompose into x and y direction, we obtain

$$g - \frac{T}{m} \cos \phi = -l \cos \phi (\dot{\phi})^2 - l \sin \phi \ddot{\phi} - A\Omega^2 \cos \Omega t$$

$$-\frac{T}{m} \sin \phi = -l \sin \phi (\dot{\phi})^2 + l \cos \phi \ddot{\phi}$$

where T is the magnitude of the force of tension on the pendulum. Multiply the first equation by $\sin \phi$, the second by $\cos \phi$. Subtract the second from the first, we obtain

$$\ddot{\phi} + \left[\frac{g}{l} + \frac{A\Omega^2}{l} \cos \Omega \right] \sin \phi = 0 \quad (1)$$

We are considering the case in which

$$\frac{A}{l} \ll 1, \quad \frac{\omega_0}{\Omega} \ll 1, \quad \left(\omega_0 = \frac{g}{l} \right).$$

B. Analysis: We guess that a solution of equation (1) is the sum of two functions: one is fast oscillating with a small magnitude, which

we denote as $\varepsilon(t)$ and the other is slowly oscillating, which we denote as $f(t)$. We write

$$\phi = f(t) + \varepsilon(t).$$

We put it into equation (1) to obtain

$$\begin{aligned} \ddot{f} + \ddot{\varepsilon} &= - \left(\frac{g}{l} + \frac{A\Omega^2}{l} \cos \Omega t \right) \sin(f + \varepsilon) \\ &\approx - \left(\frac{g}{l} + \frac{A\Omega^2}{l} \cos \Omega t \right) (\sin f + \varepsilon \cos f) \\ &= -\frac{g}{l} \sin f - \frac{A\Omega^2}{l} \cos \Omega t \sin f - \varepsilon \frac{g}{l} \cos f \\ &\quad - \varepsilon \frac{A\Omega^2}{l} \cos \Omega t \cos f \end{aligned}$$

Balancing separately larger and smaller terms we obtain:

$$\begin{aligned} \ddot{\varepsilon} &= -\frac{A\Omega^2}{l} \cos \Omega t \sin f \\ \ddot{f} &= -\frac{g}{l} \sin f - \varepsilon \frac{g}{l} \cos f \\ &\quad - \varepsilon \frac{A\Omega^2}{l} \cos \Omega t \cos f. \end{aligned} \tag{2}$$

Here we took into account that $\Omega/\omega_0 \gg 1$, $|\varepsilon(t)|$ – is small and second derivative of $\varepsilon(t)$ is large due to rapidly oscillating nature of $\varepsilon(t)$. First let us see if we can make sense of the first equation. Since $\varepsilon(t)$ changes real fast, and $f(t)$ slowly, we can then regard f as a constant when consider $\varepsilon(t)$. Regarding f as a constant, we can easily obtain a solution of the first equation as

$$\varepsilon(t) = \frac{A}{\ell} \sin f \cos \Omega t$$

This function is a rapidly oscillating function with a small amplitude A/ℓ .

Putting $\varepsilon(t)$ back into the second equation we obtain

$$\begin{aligned} \ddot{f} = & -\frac{g}{\ell} \sin f - \frac{gA}{\ell^2} \cos \Omega t \sin f \cos f \\ & - \frac{A^2 \Omega^2}{\ell^2} \cos^2 \Omega t \sin f \cos f \end{aligned} \quad (3)$$

C. Result of averaging: Equation (3) defines a 2D vector field that is time dependent. To use the analysis of 2D systems we first remove rapidly varying coefficients by means of averaging over fast oscillations. Recall that for a function $P(t)$, the average over fast oscillations is

$$\langle P \rangle = \frac{1}{T} \int_{t-T/2}^{t+T/2} P(\xi) d\xi, \quad T = \frac{2\pi}{\Omega}$$

Averaging of the equation (3) and taking into account that: (i) slowly varying function f together with derivatives is practically not changing over the period of fast oscillations; (ii) $\langle \cos \Omega t \rangle = 0$; we obtain

$$\ddot{f} + \frac{g}{\ell} \sin f + \frac{A^2 \Omega^2}{2\ell^2} \sin f \cos f = 0 \quad (4)$$

At $f = \pi$, the eigenvalues of equation (4) are

$$\lambda = i \sqrt{\frac{A\Omega^2}{2\ell^2} - \frac{g}{\ell}}.$$

Therefore $f = \pi$ is stable if

$$A\Omega > \sqrt{2g\ell}$$

or when dimensionless parameter

$$\frac{A^2\Omega^2}{2g\ell} > 1.$$

Homework: (a) Write equation (4) as a system of two first order equations and find all equilibrium solution.

(b) Compute the eigenvalues of the Jacobi matrix at all equilibrium solutions of equation (4) to determine their type.

(c) In equation (4) let $A = 0.1$, $\ell = 1$. Compute the potential function of equation (4) and use its graph to depict the phase portrait. Consider all cases regarding Ω as a parameter.