Stability of an Inverted Pendulum

Project Description

- In an ordinary pendulum, the upwards position is an unstable equilibrium point. If perturbed, it will fall back down.
- However, if the base is oscillated with sufficient frequency, the vertical position becomes stable.
- The goal of this project is to explain why this occurs using a simple model.
- Then, the model will be compared to actual experimental data. The physical pendulum is a trapezoidal rod mounted to a jigsaw.

Scientific Motivation

- It is counterintuitive for the unstable vertical position to become stable simply due to an oscillating base.

Potential Applications

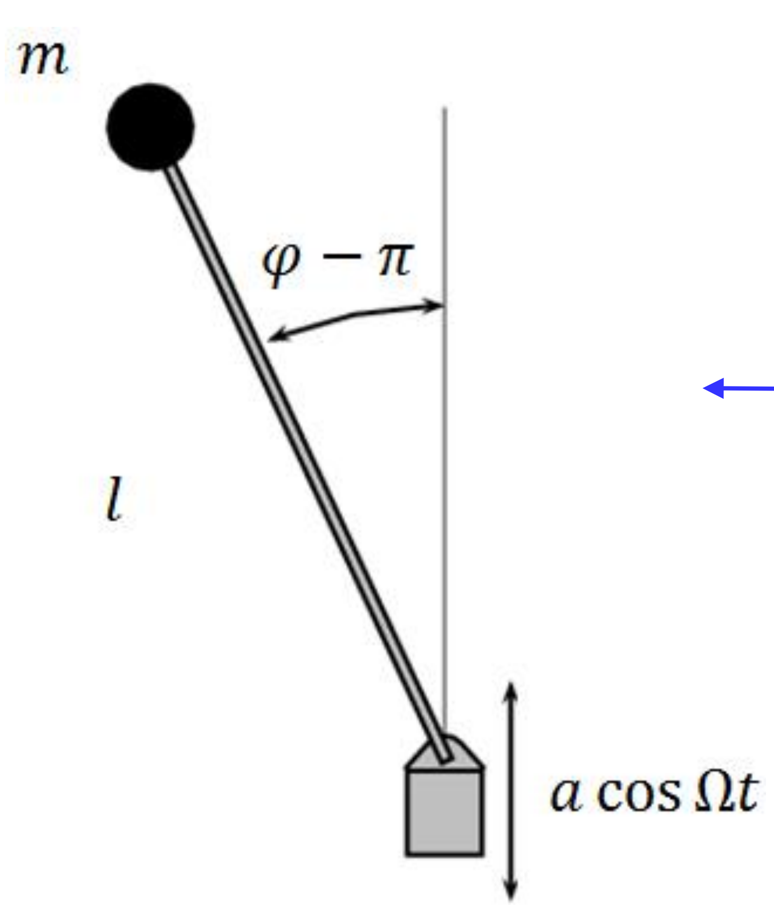
- Segways are based on a moving cart inverted pendulum.
- In biomechanics, the neck and spine can be modeled by an oscillating base inverted pendulum.

Experimental Data

Dimensions of Pendulum: trapezoidal prism
length = 28cm; base = 0.9cm x 0.9cm;
end = 0.9cm x 0.4cm;
center of mass = 12cm from base

Minimum Frequency of Stability:
206 ± 5.1 rad/s

Stability Region:
0.553 ± 0.0088 rad (31.7° ±0.5°) from vertical



Team Members:

Edward Montiel
Vicki Springmann

Methodology

Simple theoretical model:

1. The pendulum is treated as a point mass at its center of mass.
2. The equation of motion is obtained by determining the Lagrangian.
3. The vertical position becomes stable due to the interaction of scales – the fast oscillations of the base with the pendulum's slow oscillations.
4. Average and balance large terms to find an effective potential.
5. Determine the stability condition.
6. Differentiate the effective potential to determine the range of stability.

Experimental Data Collection:

1. Measure the dimensions of the pendulum. Find the center of mass.
2. Using a stroboscope, measure the minimum frequency of stability of the vertical position.
3. For a constant frequency, measure the maximum displacement from the center which remains stable. Use this to determine the stability region.

Simple Model Results

1. The minimum frequency for stability is Ω=164.6 rad/s.
2. The range of stability is 0.915 rad (52.4°) from vertical.
3. Discrepancies between model and experiment are likely because the pendulum is a bar, not a disk at the end of a thin rod.

Variable Definitions:
a: amplitude of oscillation of base
Ω: frequency of oscillation of base
φ: displacement angle of pendulum from downwards position.

g: acceleration due to Earth's gravity
l: distance to pendulum's center of mass
m: mass of the pendulum

Theory:

Lagrangian: (total derivatives, functions of only t excluded)

$$\mathcal{L} = \frac{ml^2}{2}\dot{\varphi}^2 + la\Omega^2 m \cos\Omega t \cos\varphi + mgl\cos\varphi$$

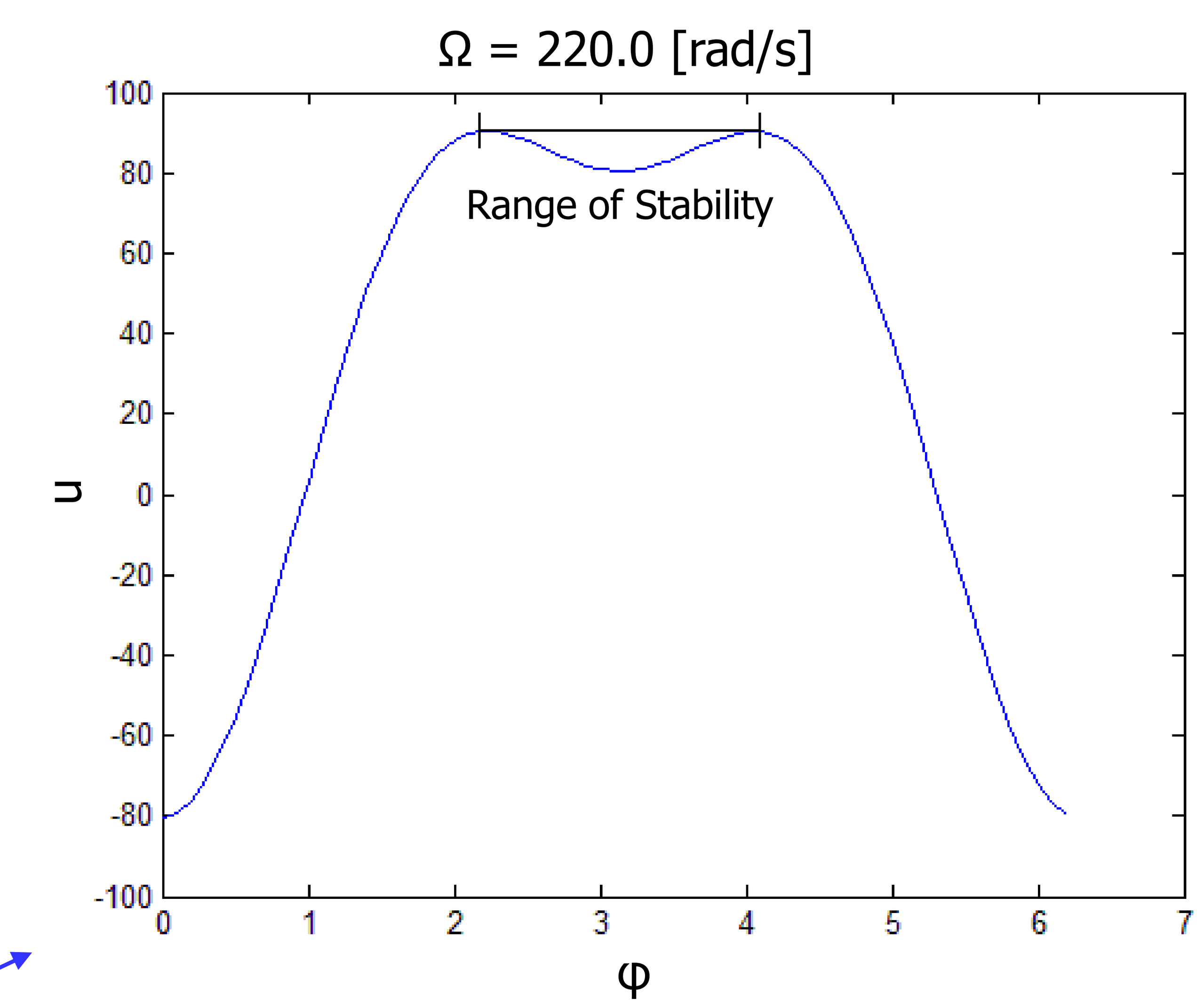
Equation of Motion:

$$\ddot{\varphi} + \left(\frac{g}{l} + \frac{a\Omega^2}{l}\cos\Omega t\right)\sin\varphi = 0$$

Effective Potential:

$$u_{eff} = \frac{g}{l}\left(-\cos\varphi + \frac{a^2\Omega^2}{4gl}\sin^2\varphi\right)$$

Stability Condition: $\frac{a\Omega}{\sqrt{2gl}} > 1$



Effective potential of the pendulum for a base oscillation frequency Ω=220.0 rad/s. Because the frequency is above the minimum required for stability, φ=π is a minimum.

References

Landau L. D. & Lifshitz E. M., *Mechanics*, (Pergamon, NY, 1960) pp 93-95. Motion in a rapidly oscillating field

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