The Stability of an Inverted Pendulum

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Abstract:

The upwards position of a simple pendulum is an unstable point. However, if a high frequency oscillation is applied to the base, the vertical position becomes stable. The stabilization of the vertical position with high driving frequencies has a well-known solution from stability analysis with an effective potential [3]. In our project, we studied the stability of the pendulum for arbitrary drive angles. The model is compared to actual experimental data from a physical pendulum where an aluminum rod attached to a jigsaw is the inverted pendulum.

Introduction

The model of the simple pendulum problem is a well-studied dynamical system. The governing equation for the simple pendulum is $\frac{d^2\theta}{dt^2} + \frac{g}{l} sin\theta = 0$, where g is gravitational acceleration, l is the length of the pendulum, and θ is the angular displacement about downward vertical. When the pendulum is given no energy, it will statically hang down at $\theta = 0$. If the pendulum is given a little bit of kinetic energy, the pendulum will oscillate about $\theta = 0$. Theoretically, if the pendulum contains a specific amount of energy, the pendulum can stay still at or asymptotically approach the standing up position at $\theta = \pi$. If the pendulum contains even more energy, the pendulum will continuously swing around and around in either the clockwise or counter-clockwise direction.

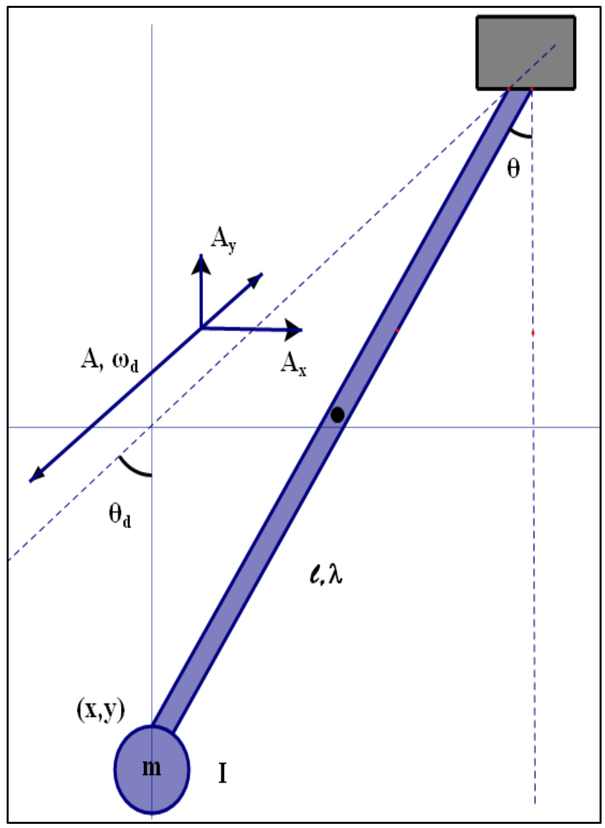
The inverted pendulum system consists of a pendulum with its center of mass above its pivot point which is mounted on the base with a pin. However, a pendulum at this position is unstable – even a small disturbance can cause it to swing down. In our model, we applied a sinusoidal oscillation to the pin at some driving angle θ_d measured from the downward position. It is well-known that for vertical oscillations, if the oscillation is strong enough and/or fast enough, the inverted position of the pendulum will become stable. The goal of our study is to analyze and simulate the stabilization of the inverted pendulum.

In our study, we used methods from Lagrangian mechanics and the Euler- Lagrange equation to derive the equation of motion. Numerical analysis was used to find the stability angle using MATLAB's ODE 45 method. MATLAB solves the derived governing equation with three driving angles of the pendulum. Then we applied averaging techniques in order to determine an effective potential energy of the driven pendulum. Applying the derivative tests for stability using the effective potential, we analyzed the stability of the pendulum for different θ. A bifurcation diagram was developed base on this analysis. We developed phase portraits for the driven pendulum based on the effective potential and developed graphical dynamic manifolds to represent the behavior of from these phase portraits. The experimental data was collected using an aluminum rod attached to a jigsaw. Both the minimum frequency for stability and the region of stability were experimentally measured.

Theory

Consider a two-dimensional model for the driven pendulum where the pendulum rotates in the same plane as the driving oscillations. In the figure to the right, gravity is assumed to be acting downward. The horizontal x-axis is defined with positive to the right and the vertical y-axis is defined with positive pointing upward. Angles are measured from downward (0^o) with clockwise being positive. The table below describes the variables we used.

Symbol	Meaning
t	Time
$x\,,y$	Position of pendulum's center of mass
g	Acceleration due to gravity
θ	Angle of deviation of pendulum
m	Mass of pendulum
l	Distance from center of mass to base
I_o	Rotational inertia about base
λ	Effective length of pendulum $\left(\frac{I_o}{ml^2}\right)$
ω_o	Natural frequency of pendulum $\left(\sqrt{\frac{g}{l}}\right)$
θ_d	Drive angle of the base
ω_d	Drive frequency
$A_x\,,A_y$	x- and y- amplitudes of the driving



We applied principles from Lagrangian mechanics and the Euler-Lagrange equation to the driven pendulum in order to find the governing equation of motion. The Lagrangian $\mathcal{L}$ for Newtonian mechanics is the difference between the kinetic T and potential V energies:

(1) $$\mathcal{L} = \mathrm{T} - \mathrm{V}$$

The equation of motion is derived from the Euler-Lagrange equation:

(2) $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

The kinetic energy of the pendulum was decomposed as the sum of rotational T_θ and translational kinetic energies T_x and T_y [2]. The potential energy is the product of the weight mg of the pendulum with the height y of its center of mass.

(3) $$T = T_x + T_y + T_\theta$$

(4) $$T_x = \frac{1}{2}m\left(l^2\dot{\theta}^2\cos^2\theta + A_x^2\omega_d^2\sin^2(\omega_d t)\right) - 2A_x\omega_d l\dot{\theta}\sin(\omega_d t)\cos\theta)$$

(5) $$T_y = \frac{1}{2}m\left(l^2\dot{\theta}^2\sin^2\theta + A_y^2\omega_d^2\sin^2(\omega_d t)\right) + 2A_y\omega_d l\dot{\theta}\sin(\omega_d t)\sin\theta)$$

(6) $$T_\theta = \frac{1}{2}I_o\dot{\theta}^2 - \frac{1}{2}ml^2\dot{\theta}^2$$

(7) $$V = mgy = -mg(l\cos\theta + A_y\cos(\omega_d t))$$

After solving for kinetic and potential energy using equations (3) – (7), we plugged the energies into equations (1) and (2) to find the equation of motion:

$$I_o\ddot{\theta} + mgl\sin\theta + mAl\omega_d^2(\cos\theta_d\sin\theta - \sin\theta_d\cos\theta)\cos\omega_d t = 0$$

We introduced an effective length λ of the pendulum where:

$$\lambda = \frac{I_o}{ml}$$

Making this substitution, we have our equation of motion:

(8) $$\ddot{\theta} + \frac{g}{\lambda}\sin\theta + \frac{A\omega_d^2}{\lambda}sin(\theta - \theta_d)cos(\omega_d t) = 0$$

We introduced three dimensionless parameters and an expression for the driving effects: τ is a non-dimensional time, γ is a frequency parameter, α is a length parameter and $D(\theta, \tau)$ is the expression representing effects caused by the driving.

$$\tau = \omega_d t \qquad \gamma = \frac{\omega_o^2}{\omega_d^2} \qquad \alpha = \frac{A}{\lambda}$$

τ is equal to the full periods of oscillation the base has gone through since some initial time. γ is inversely proportional to the drive frequency; smaller γ represents faster drive frequencies. α is proportional to the amplitude of the driving oscillations. Writing our equation of motion in terms of these dimensionless parameters gives us the non-dimensional form of the equation of motion:

$$\frac{d^2\theta}{d\tau^2} + \gamma \sin\theta + \alpha \sin(\theta - \theta_d)\cos\tau = 0 \quad (9)$$

Computational Analysis

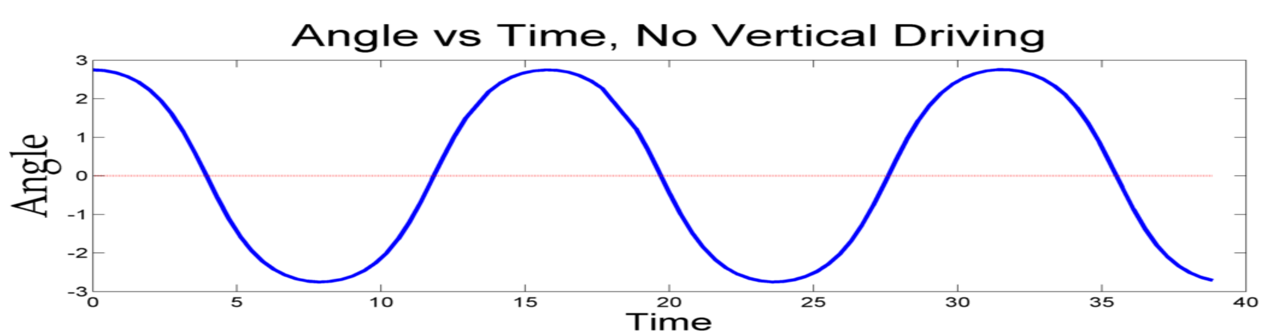
Using the derived governing equation of motion from the Lagrangian analysis, we can take a numerical look at how any particular pendulum will behave when driven at any particular angle. For the purposes of our analysis, we are using the parameters associated with our Black and Decker jigsaw, including its effective length, driving frequency, and driving amplitude. By implementing Matlab's ODE45 method, we can set the initial angle and initial velocity for the pendulum and plot the solution for the pendulum's angle vs the time passed since the pendulum's release. ODE45 uses a Runge-Kutta variable step method to solve our differential equation, which Matlab then plots. We decided to use this method for three different driving angles, as follows:

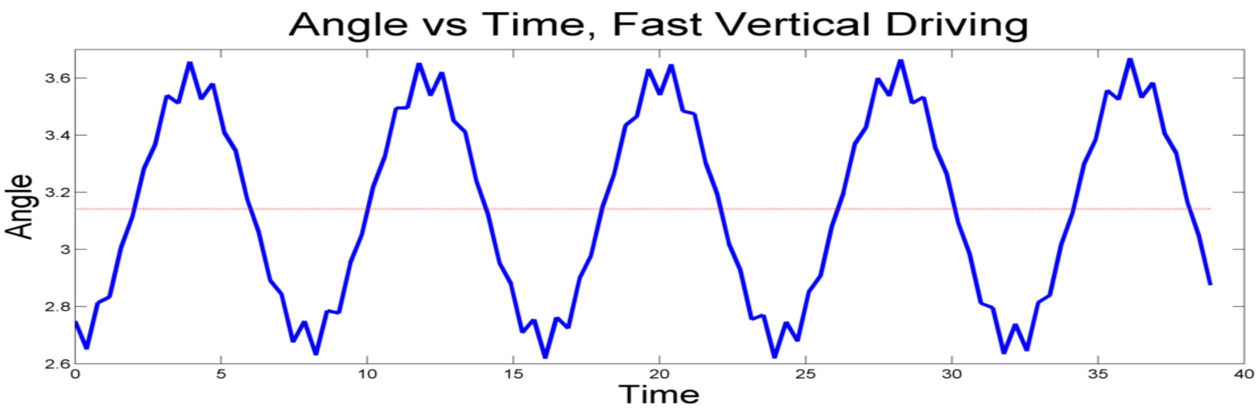
$$\theta_d = 0; \textit{Drives about the vertical}$$

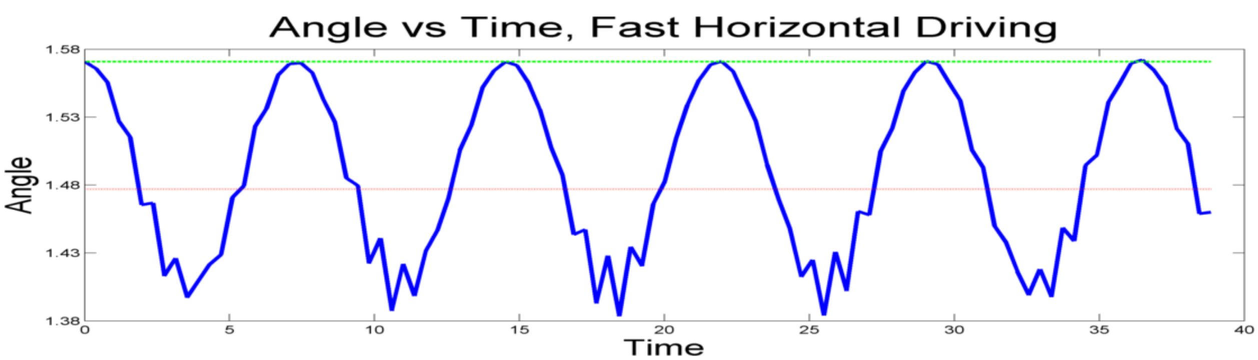
$$\theta_d = \frac{\pi}{4}; \textit{Drives about the diagonal}$$

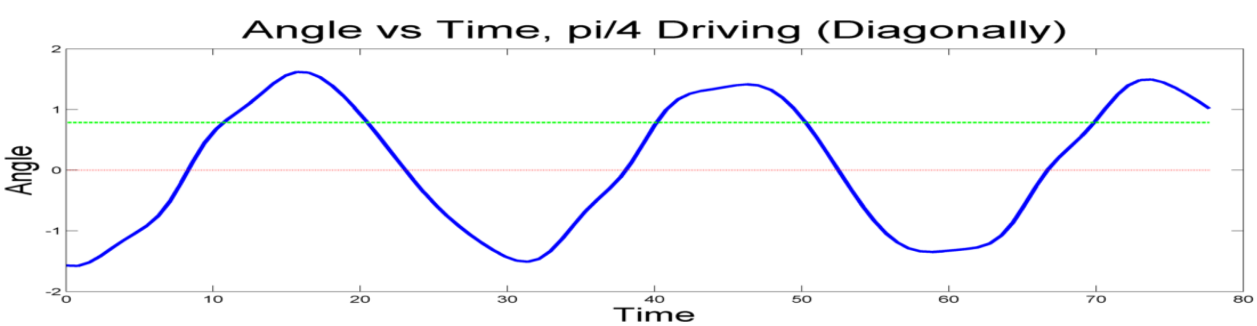
$$\theta_d = \frac{\pi}{2}; \textit{Drives about the horizontal}$$

For the sake of avoiding redundancy, symmetrical angles have been disregarded. By setting appropriate initial conditions, we have observed oscillatory behavior for each driving angle, indicating the stability points therein. The following are figures taken from our Matlab simulations. The green lines represent driving angles while red lines represent determined average angles found from the analysis.









As expected, the stability point for the horizontal driving angle is located below the horizontal, due to gravity. Notably, the diagonal driving angle has produced differing results from the other two driving angles. It oscillates about 0, as expected, but with lengthy and wide oscillations. This is most likely due to the volatility of the equilibrium angle as the angle slightly increases from $\frac{\pi}{4}$. Further analysis and experimentation will test the accuracy of this model with the actual behavior of the jigsaw.

Averaging Methods & Effective Potential

We applied averaging techniques and found the average or effective potential energy of the driven pendulum. The effective potential U_{eff} will be a measure of the average potential energy the pendulum has at certain pendulum angles of deviation θ [1]. We did this by averaging the potential energy over a period of the driving oscillation. The first step is to separate the slow and fast components of the pendulum. ϕ is the slower angle of the pendulum while ξ is the rapid oscillations of the base. $F(\theta)$ is the slower force of gravity while $f(\theta, \tau)$ is the rapidly oscillating force from the driving.

$$I_o\ddot{\theta} = I_o(\ddot{\phi} + \ddot{\xi}) = (F(\theta) + f(\theta, t)) \tag{10}$$

Considering ξ as the difference between ϕ and θ, we can do a first-order Taylor approximation to (10). Making the assumption that ξ is insignificantly small and ω_d is significantly large, we can ignore negligible terms. After averaging the equation (10) over the period of oscillations of the base, we derived equation (11).

$$I_o\ddot{\phi} \cong F(\phi) + \langle \frac{df}{d\theta}(\phi, t)\xi \rangle \tag{11}$$

Here is where we see the interaction of the two scales: large-and-slow and small-and-fast. Usually, small terms are insignificant and have little effect on the macroscopic behavior of a system. But, in the case of the driven pendulum, we see that the swing of the pendulum is related to the time-derivatives of the fast scale. The derivatives of the fast scale are proportional to the square of the driving frequency ω_d^2.

Now, let's define the U_{eff}:

$$I_o\ddot{\phi} = \frac{-dU_{eff}}{d\phi} \tag{12}$$

Writing the forces using the terms from the equation of motion, we have:

$$F(\theta) = I_o\omega_d^2\gamma\sin\theta \qquad\qquad f(\theta, t) = -I_o\omega_d^2\alpha\, D(\theta, t)$$

Using these expressions for force and plugging them into (11) and (12), we solved for U_{eff}:

$$U_{eff} = I_o\omega_d^2\left(-\gamma\cos\theta + \frac{\alpha^2}{4}(\cos^2\theta_d\sin^2\theta + \sin^2\theta_d\cos^2\theta)\right) \tag{13}$$

With U_{eff}, we analyzed the stability of the pendulum for different θ. Equilibrium angles θ_{eq} are predicted to occur where the derivative of (13) evaluates to zero:

(14) $$\frac{dU_{eff}}{d\theta} = 0 = I_o\omega_d^2 \sin\theta \left(\frac{\alpha^2}{2}\cos 2\theta_d \cos\theta_{eq} + \gamma\right)$$

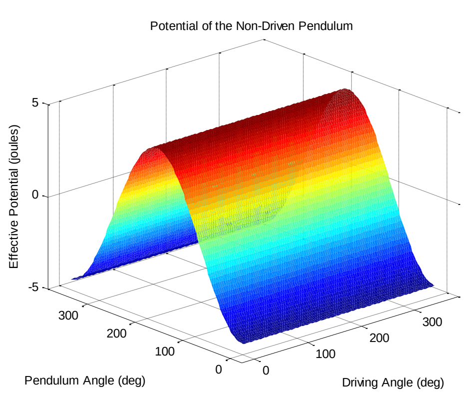
There are three different types of equilibrium angles: hanging down ($\theta_{eq} = 0$), standing up ($\theta_{eq} = \pi$), or leaning to the side ($\theta_{eq} = \pm \arccos\left(\frac{2\gamma}{\alpha^2 \cos 2\theta_d}\right)$). To determine whether these equilibrium angles are stable or unstable, we looked to see if the second derivative of (13) at these angles is positive – meaning stable – or negative – meaning unstable.

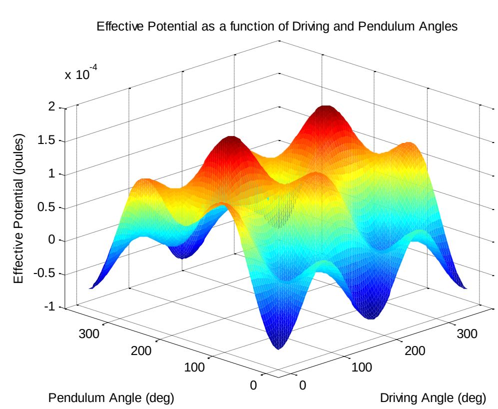
(15) $$\frac{d^2U_{eff}}{d\theta^2} = I_o\omega_d^2 \left(\frac{\alpha^2}{2}\cos 2\theta_d \cos 2\theta_{eq} + \gamma \cos\theta_{eq}\right)$$

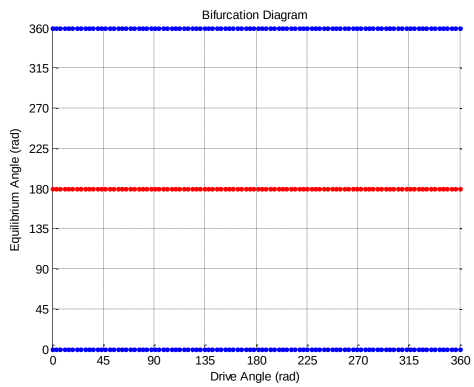
The conditions for stability are summarize in the table below for each equilibrium:

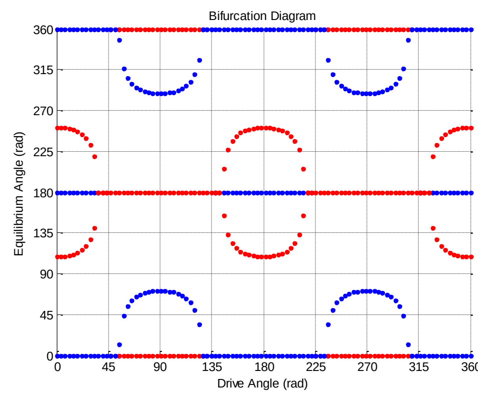
Equilibrium (θ_{eq})		Stable	Unstable
$\theta_{eq} = 0$		$\gamma > \frac{-\alpha^2}{2}\cos 2\theta_d$	$\gamma < \frac{-\alpha^2}{2}\cos 2\theta_d$
$\theta_{eq} = \pi$		$\gamma < \frac{\alpha^2}{2}\cos 2\theta_d$	$\gamma > \frac{\alpha^2}{2}\cos 2\theta_d$
$\theta_{eq} = \pm\, arccos\left(\frac{-2\gamma}{\alpha^2\, cos\, 2\theta_d}\right)$	$\cos 2\theta_d > 0$	$\gamma^2 > \frac{\alpha^4 cos^2(2\theta_d)}{4}$	
	$\cos 2\theta_d = 0$	Always stable	
	$\cos 2\theta_d < 0$	$\gamma^2 < \frac{\alpha^4 cos^2(2\theta_d)}{4}$	

Using (13), we used MatLab to make surface plots of U_{eff} as a function of drive angle θ_d and pendulum angle θ. Then, we also used the stability conditions summarized in the last table to create a bifurcation diagram for the pendulum. Surface plots and bifurcation diagrams are shown below for the undriven – to the left – and driven – on the right – pendulums.





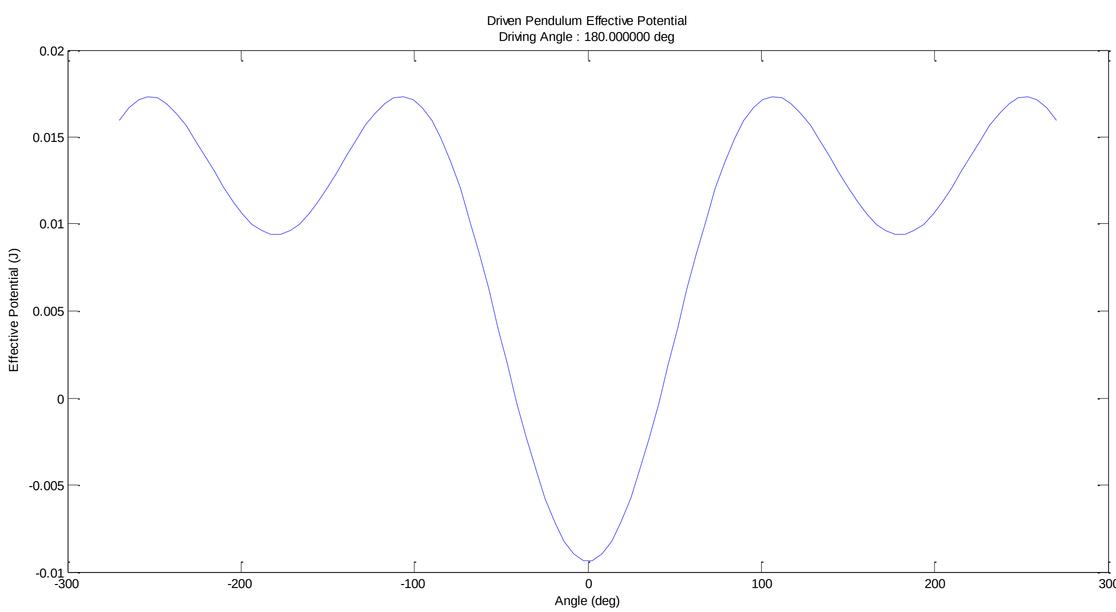




- stable
- unstable

Phase Portraits & Dynamic Manifolds

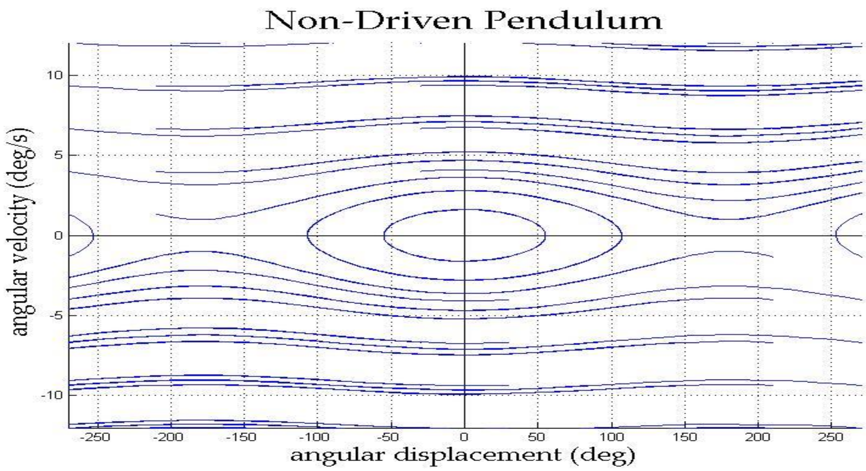
The phase portrait of the pendulum is a plot of angular velocity $\dot{\theta}$ versus angular displacement θ. The plots of the phase portraits are made by modeling the motion of the pendulum as a ball rolling along a 2D contour laid out by the effective potential of the pendulum. Consider the following plot of U_{eff}versus θ for vertical driving angle $\theta_d = \pi$.

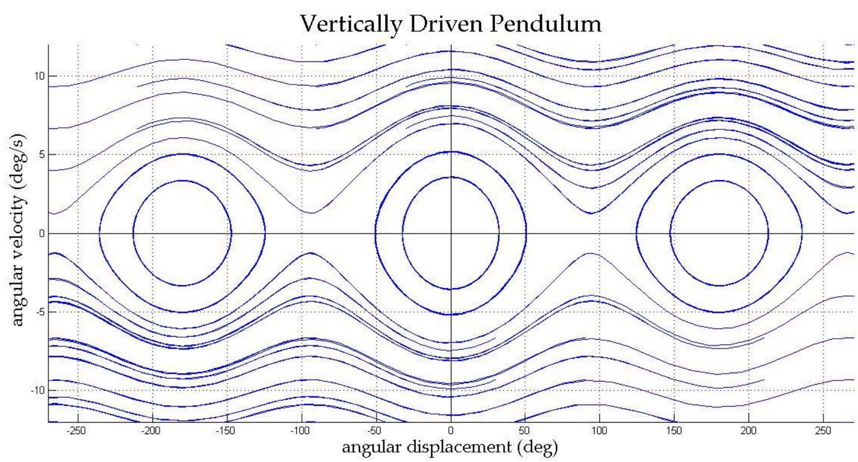


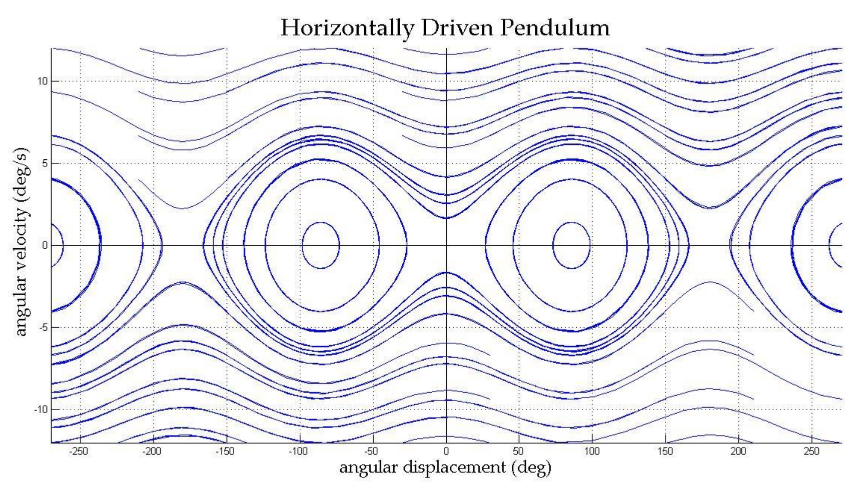
If we consider a ball rolling along the top of the curve by a fictional downward force of gravity, then we find the acceleration of the ball to be:

$$\ddot{\theta} = -G \sin\left(\arctan\left(\frac{dU_{eff}}{d\theta}\right)\right) \quad (16)$$

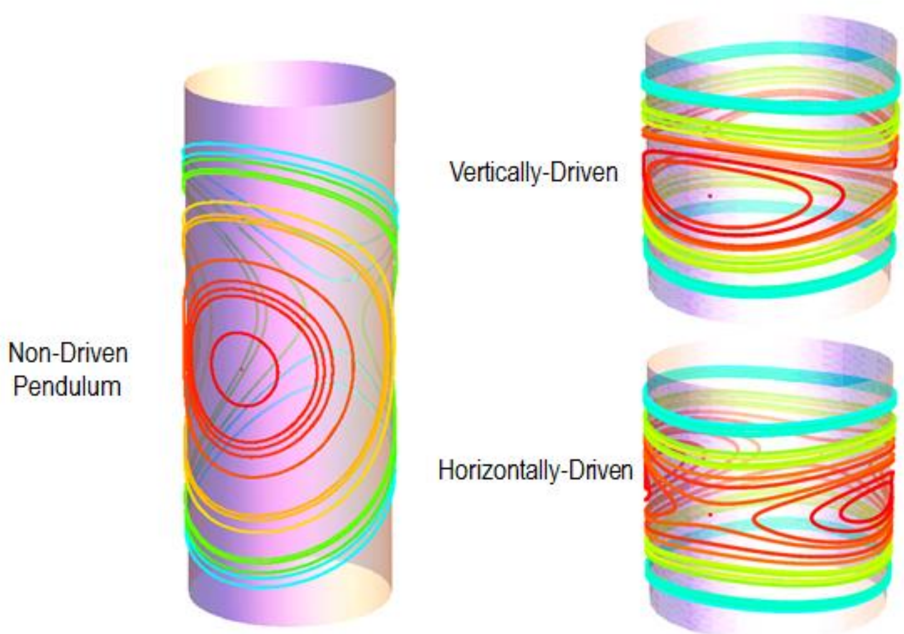
Where G is related to some fictitious force of gravity related to the resolution of the phase portrait. Using (16), we created the following phase portraits for the horizontally and vertically driven pendulums. We also show the phase portrait for the simple non-driven pendulum.







If we consider the periodic nature of the phase portrait, we can imagine taking the ends of the portrait at $\theta = -\pi$ and $\theta = \pi$ and wrapping them around and gluing them together to form a cylindrical surface. Also, if we map each contour on this surface to a certain energy level, then we will have a surface that graphically represents the motion of the pendulum which is mapped to the energy level corresponding to that behavior. This is our dynamic manifold. Below we show dynamic manifolds made in Mathematica for the simple non-driven pendulum, horizontally, and vertically driven pendulums.



Experimentation

We tested our theory with a Black and Decker JS515 Jigsaw. Attached to the jigsaw was a metal rod to act as the pendulum. Due to time and money restraints, we were only able to study this phenomenon with video clips from previous years. With a high speed camera, videos were shot at 6000 frames per second and played at 29.9 frames per second, so 200.669 seconds in the video is actually 1 second in real life. For our experiment, we used three different videos: two with the pendulum being driven vertically at different frequencies and one horizontally. Unfortunately we weren't given a video with the pendulum being driven with a diagonal angle. Using frame-by-frame

analysis and on-screen measurements, we were able to determine the stability angles for each video. Our results are summarized in the following table.

Measurement (driving)	Theoretical	Experimental
Vertical Stability Angle	π (straight up)	π
Horizontal Stability Angle	1.477 radians	~1.405 radians
Diagonal Stability Angle	0 (straight down)	-
Vertical Frequency	314.16 dps	300.3 dps
Horizontal Frequency	314.16 dps	453 dps

When we compare our experimental results with our theoretical results, we find our stability angles coincide with what we expected them to be. But when we compare our experimental frequencies to the theoretical ones, they seem to be way off. In reality they are off by about four degrees of error (4.87%). This is most likely due to the fact that we used data about the jigsaw from the owner's manual and each jigsaw is slightly different because of usage and the modifications made to the jigsaw. The error may also come from our assumption that friction is insignificant in our model.

Conclusion

In our work, we analyzed the stability of the inverted pendulum caused by the oscillating base and demonstrated the case through computational analysis, averaging techniques, and experimentation. We extended our analysis to sinusoidally oscillating bases at any arbitrary constant driving angle. After analyzing the stability of the pendulum for arbitrary drive angles, we applied the same three methods – computational analysis, averaging techniques, and experimentation – to the model. The theoretical predictions proposed by the averaging techniques are verified by the computational analysis which is further supported by the experimentation.

Future Work

So far, the results of our analysis have applied to a driven pendulum with very high driving frequencies relative to the natural frequency of pendulum: $\omega_d \gg \omega_o$ or $\gamma \ll 1$. We haven't answered the question: "How fast or how strong do we have to drive the pendulum in order to stabilize these nonintuitive angles?" In future work, our group hopes to find these critical driving frequencies – ω_c and γ_c – and critical driving amplitudes – A_c and α_c – which separate realms of behavior of the pendulum.

We have also begun work in analyzing the behavior of driven multiple-bar pendulums – bars linked end-to-end with one of the bars in the chain fixed to a driving base. For example, a two-bar pendulum is commonly referred to as a double-pendulum.

Acknowledgements

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