Stability of Lagrange Points: The James Webb Space Telescope

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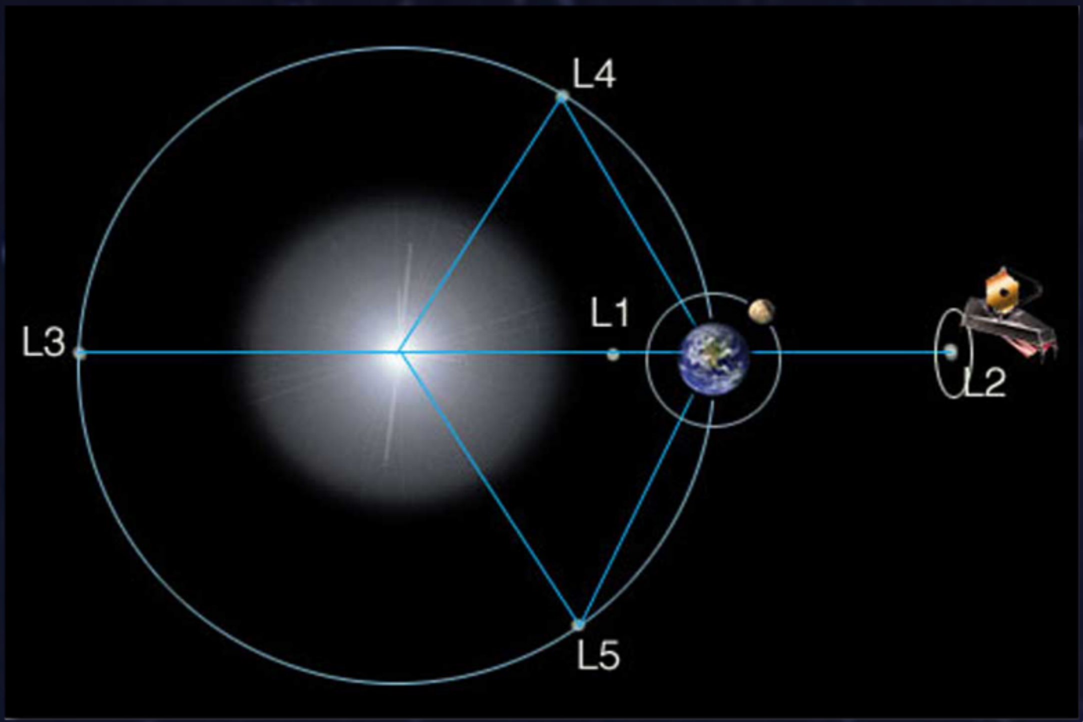
Lagrange Points: Refresher



Locations of the Lagrange points

$$L1: \left(R\left[1-\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right], 0\right)$$

$$L2: \left(R\left[1+\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right], 0\right)$$

$$L3: \left(-R\left[1+\frac{5\alpha}{12}\right], 0\right)$$

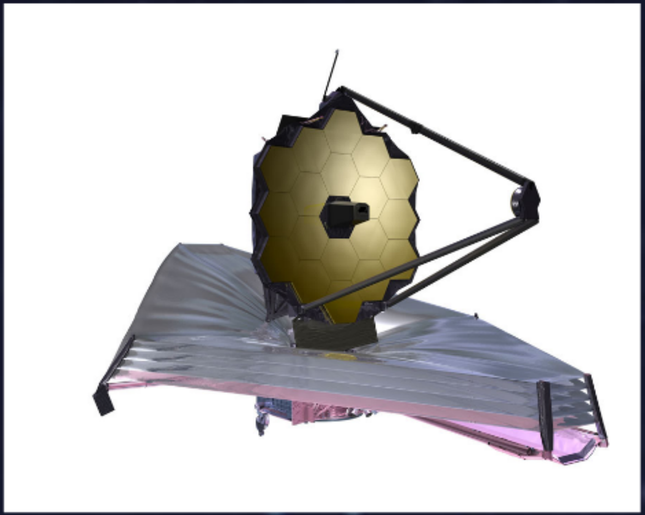
$$\alpha = \frac{M_2}{M_1+M_2}$$

$$L4: \quad \left(\frac{R}{2}\left(\frac{M_1-M_2}{M_1+M_2}\right), \frac{\sqrt{3}}{2}R\right),$$

$$L5: \quad \left(\frac{R}{2}\left(\frac{M_1-M_2}{M_1+M_2}\right), -\frac{\sqrt{3}}{2}R\right).$$

James Webb Space Telescope

- Infrared space telescope
- Successor to Hubble Space Telescope
- Investigating cosmological questions
- **Located at the second Lagrange Point (L2)**



Analysis of Stability: Analytically

- Normally: Use force to find effective potential
  - Examine the shape of potential to assess stability
- However, we have a velocity dependent force
  - Co-rotating frame used to solve equations
  - Coriolis force which depends on velocity

Method for Analysis

- Continue to use co-rotating frame
- Linearize our equations of motion
  - Solve for small departures from equilibrium for both position and velocity

$$U_\Omega = U - \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2}(\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r})$$

Equations

- Begin by writing:

$$x = x_i + \partial x, \qquad v_x = \partial v_x$$
$$y = y_i + \partial y, \qquad v_y = \partial v_y$$

- Linearized equations of motion become:

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta y \\ \delta v_x \\ \delta v_y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{d^2 U_\Omega}{dx^2} & \frac{d^2 U_\Omega}{dxdy} & 0 & 2\Omega \\ \frac{d^2 U_\Omega}{dydx} & \frac{d^2 U_\Omega}{dy^2} & -2\Omega & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta v_x \\ \delta v_y \end{pmatrix}.$$

- Eigenvalues describe stability!

L1 and L2

L1:

$$\frac{\partial^2 U_\Omega}{\partial x^2} = -9\Omega^2$$

$$\frac{\partial^2 U_\Omega}{\partial x \partial y} = 0$$

$$\frac{\partial^2 U_\Omega}{\partial y \partial x} = 0$$

$$\frac{\partial^2 U_\Omega}{\partial y^2} = 3\Omega^2$$

L2:

$$\frac{\partial^2 U_\Omega}{\partial x^2} = 9\Omega^2$$

$$\frac{\partial^2 U_\Omega}{\partial x \partial y} = 0$$

$$\frac{\partial^2 U_\Omega}{\partial y \partial x} = 0$$

$$\frac{\partial^2 U_\Omega}{\partial y^2} = -3\Omega^2$$

Eigenvalues:

$$\lambda_{1,2} = \pm\sqrt{1 + 2\sqrt{7}}$$

- One of the eigenvalues is real and positive.
- L1 and L2 are dynamically unstable. If an object is perturbed slightly from equilibrium, it will continue to move away.

L3

L3:

$$\frac{\partial^2 U_\Omega}{\partial x^2} = -3\Omega^2$$

$$\frac{\partial^2 U_\Omega}{\partial x \partial y} = 0$$

$$\frac{\partial^2 U_\Omega}{\partial y \partial x} = 0$$

$$\frac{\partial^2 U_\Omega}{\partial y^2} = \frac{7\alpha}{8\beta}\Omega^2$$

Eigenvalues:

$$\lambda_{1,2} = \pm\Omega\sqrt{\frac{3\alpha}{8\beta}}$$

- Again there is one real, positive eigenvalue
- L3 is dynamically unstable. But the effect is less pronounced than at L1 or L2, meaning the object will take longer to drift away from equilibrium.

L4 & L5

L4:

$$\frac{\partial^2 U_\Omega}{\partial x^2} = \frac{3}{4}\Omega^2$$

$$\frac{\partial^2 U_\Omega}{\partial x \partial y} = \frac{3\sqrt{3}}{4}(\beta - \alpha)\Omega^2$$

$$\frac{\partial^2 U_\Omega}{\partial y \partial x} = \frac{3\sqrt{3}}{4}(\beta - \alpha)\Omega^2$$

$$\frac{\partial^2 U_\Omega}{\partial y^2} = \frac{9}{4}\Omega^2$$

L5:

$$\frac{\partial^2 U_\Omega}{\partial x^2} = \frac{3}{4}\Omega^2$$

$$\frac{\partial^2 U_\Omega}{\partial x \partial y} = \frac{3\sqrt{3}}{4}(\alpha - \beta)\Omega^2$$

$$\frac{\partial^2 U_\Omega}{\partial y \partial x} = \frac{3\sqrt{3}}{4}(\alpha - \beta)\Omega^2$$

$$\frac{\partial^2 U_\Omega}{\partial y^2} = \frac{9}{4}\Omega^2$$

Eigenvalues:

$$\lambda_{1,2} = \pm i \frac{\Omega}{2}\sqrt{2 - \sqrt{27(\beta - \alpha)^2 - 23}}$$

- Eigenvalues of the linearized matrix are all imaginary in the Earth-Sun system
- L4 and L5 are stable equilibrium points

Results for the Stability

L1 and L2, L3:

- saddle points (one real, positive eigenvalue)
- dynamically unstable
- L3 is closer to stable (much smaller eigenvalue)

L4 and L5:

- stable (eigenvalues have no real components)
-

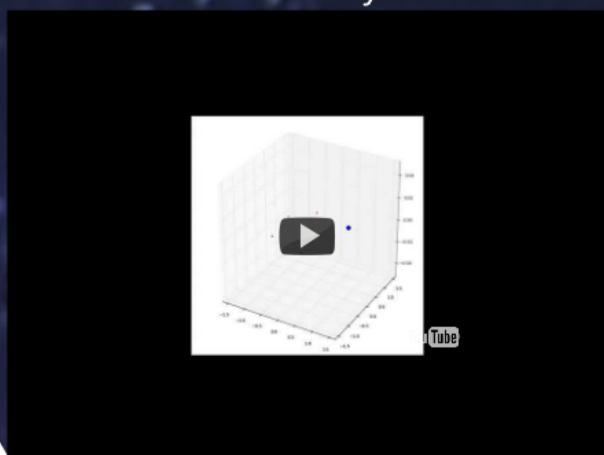
Computational Simulation

- 3D Nbody in C++ and Python
- 4th Order Hermite Integrator (Predictor-Corrector)
- Position and its first 3 time derivations were computed about every 1.75 minutes in simulation world

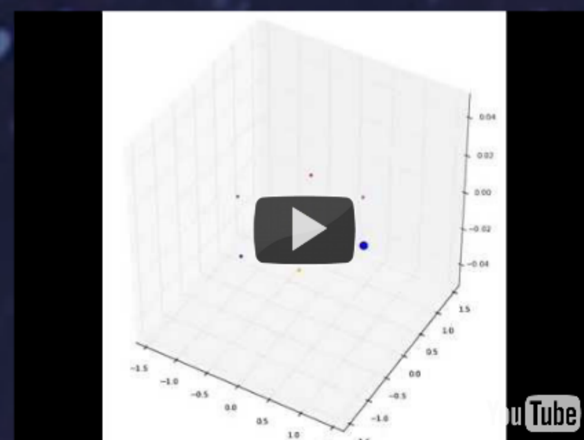
Computational Videos

10 year orbital evolution of various bodies randomly perturbed from equilibrium

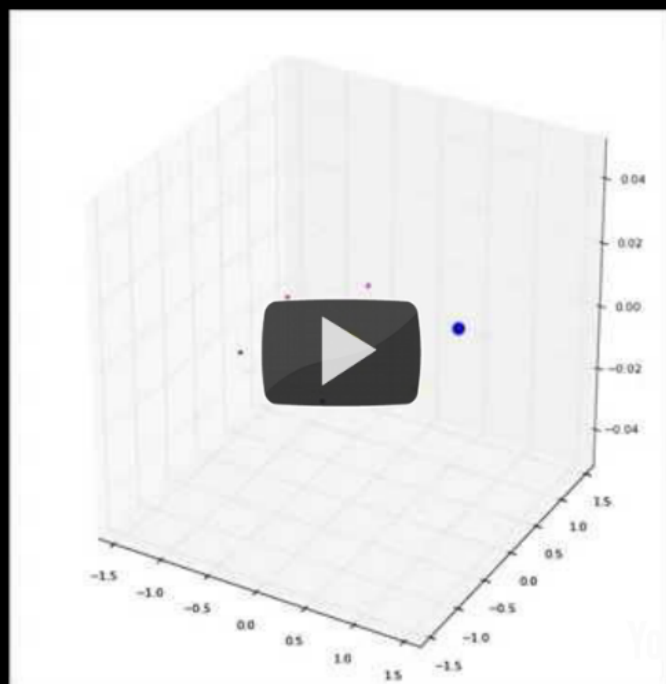
Orbits in Stationary Frame



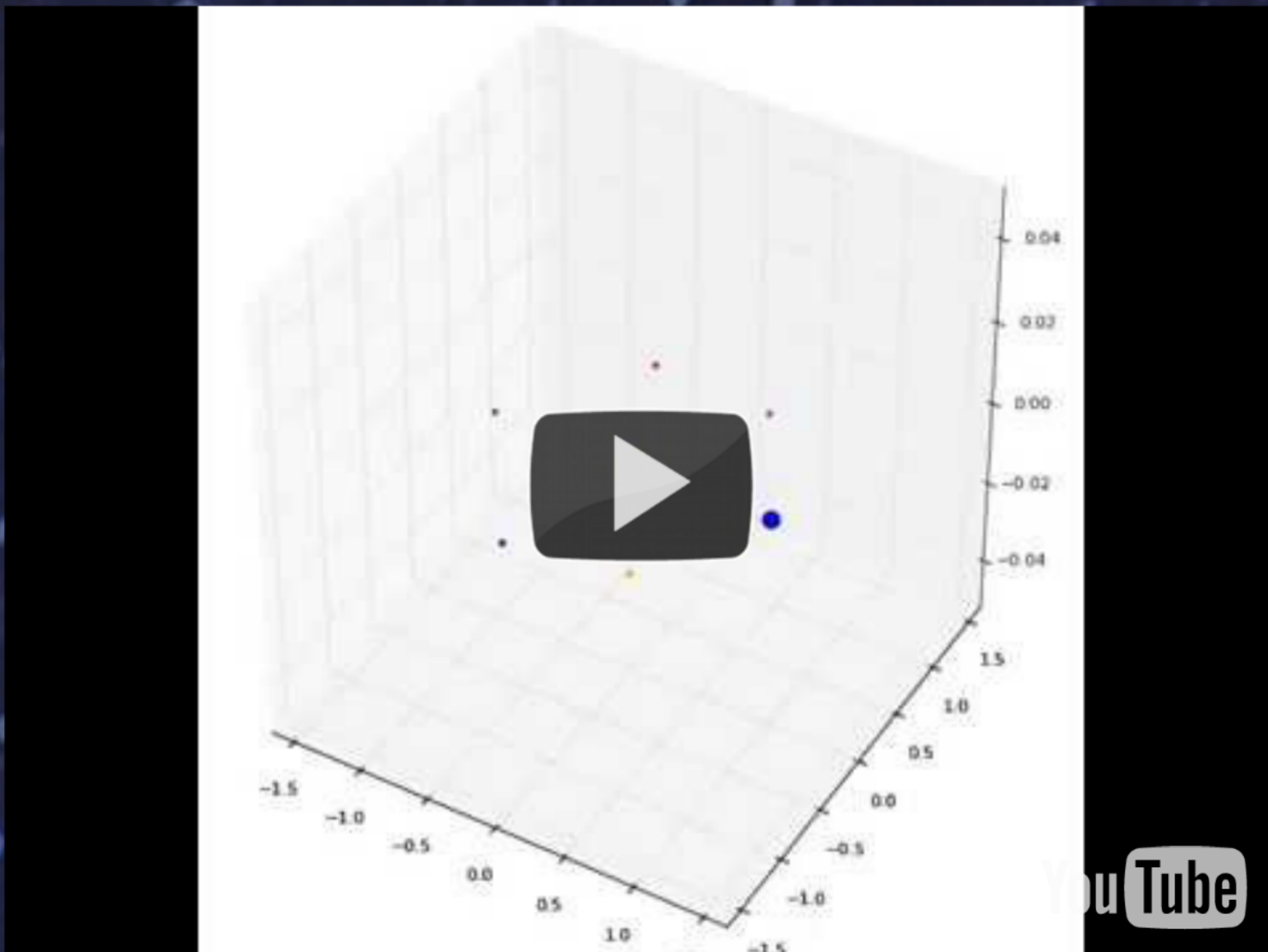
Orbits in Co-rotating Frame



Orbits in Stationary Frame



Orbits in Co-rotating Frame

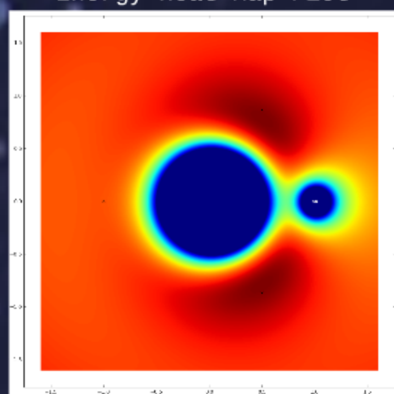


Computational Results

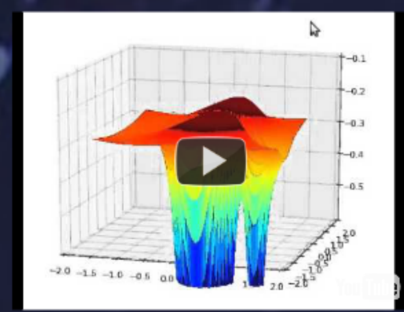
The stable points are L4 and L5.

Point	Drift Time (Computational)	Drift Time (Analytical)
L1	24.9 Days	~23 Days
L2	24.4 Days	~23 Days
L3	43.12 Years	~150 Years

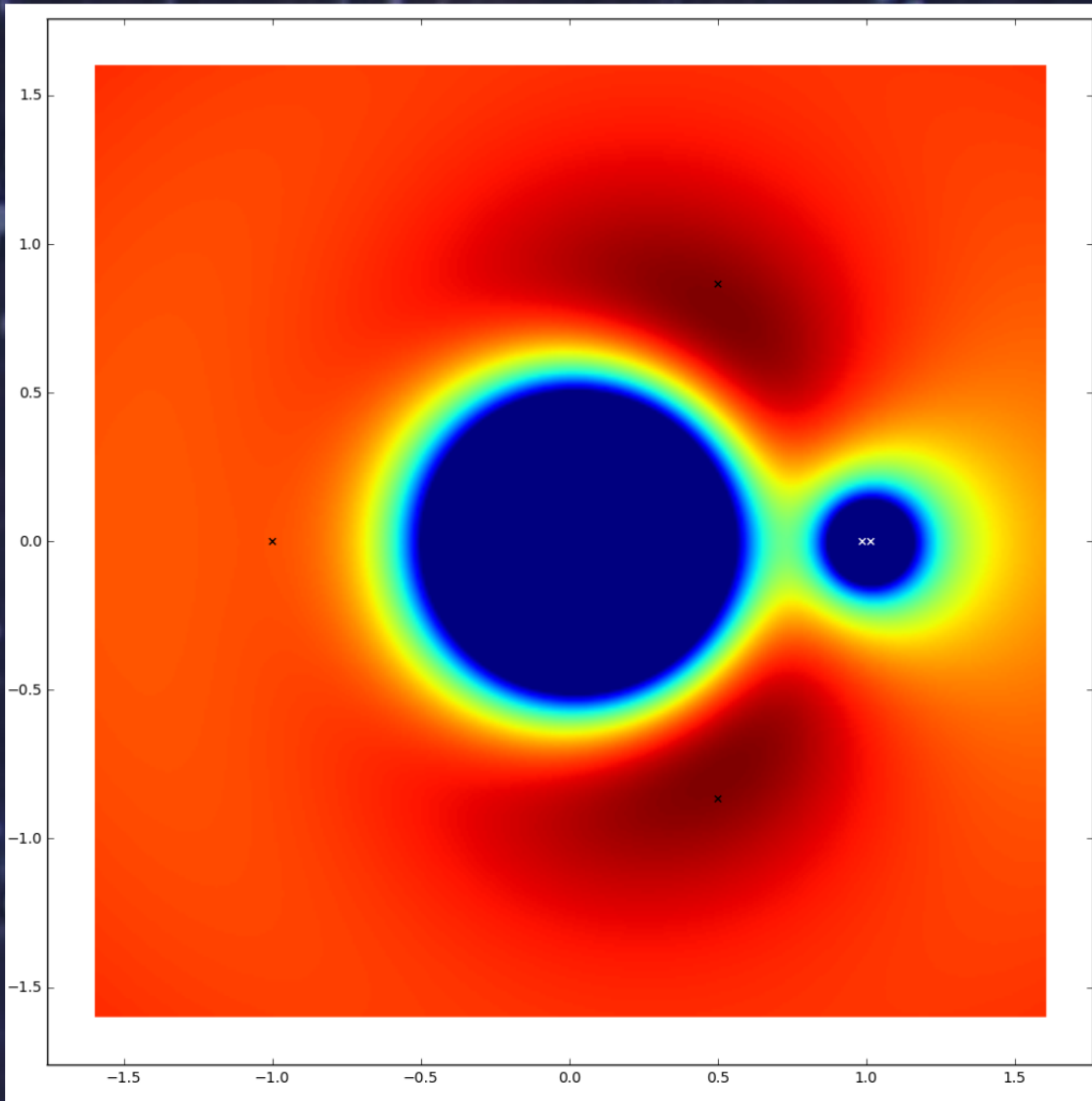
Energy Heat Map Plot



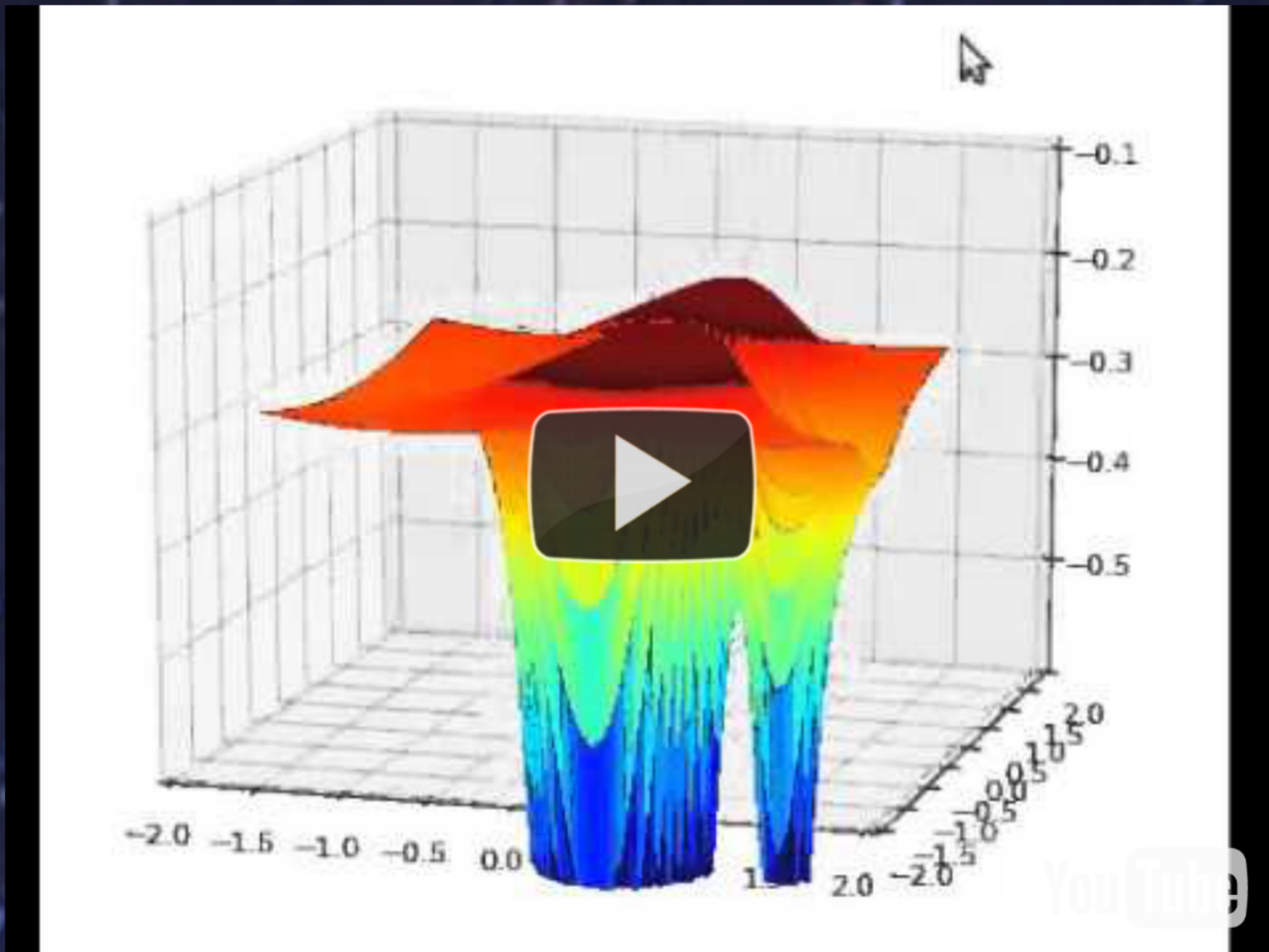
3D Energy Surface



Energy Heat Map Plot



3D Energy Surface



Summary of Lagrange Point Stability

Point	Location	Stability	Deviation after one month
L1	$\left(R\left[1-\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right],0\right)$	Saddle	0.007315 AU
L2	$\left(R\left[1+\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right],0\right)$	Saddle	0.007285 AU
L3	$\left(-R\left[1-\left(\frac{5\alpha}{12}\right)^{\frac{1}{3}}\right],0\right)$	Saddle	0.000169 AU
L4	$\left(\frac{R}{2}\left[\frac{M_1-M_2}{M_1+M_2}\right],\frac{\sqrt{3}}{2}R\right)$	Stable Due to coriolis force	0.000166 AU
L5	$\left(\frac{R}{2}\left[\frac{M_1-M_2}{M_1+M_2}\right],-\frac{\sqrt{3}}{2}R\right)$	Stable Due to coriolis force	0.000169 AU

$$\alpha = \frac{M_2}{M_1+M_2}$$

Implications for James Webb Space Telescope

- Goal is to have JWST active from 5-10 years
- As Shown Above
  - L2 stable only for about 1 months
- JWST will orbit in an elliptical orbit around the L2 point
- Need engines to keep it in place
  - Engines will have to work every few weeks

Was L2 a good Location for JWST

- Good Shielding from Earth and Sun
- Good Location for Data transfer
- Easier to reach for launch

However, the drift is large and needs to be taken into account very often

References

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