Bubble Dynamics in a Vibrating Liquid

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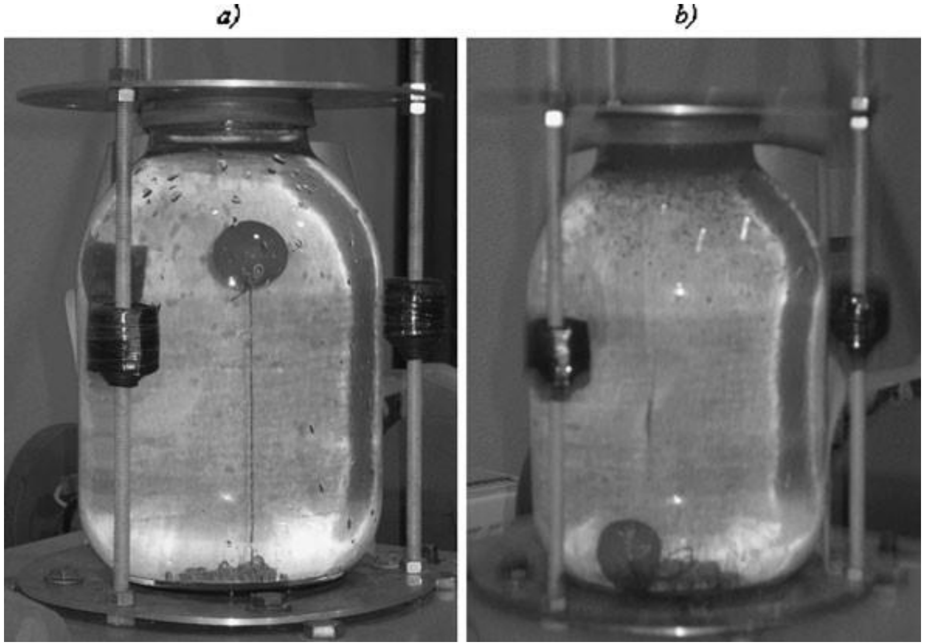
Observation

- Bubbles can sink in a vibrating liquid
- The vibration can come from the liquid being directly agitated or agitation of its container
- Fluid density and pressure, bubble depth and amount of vibration are all key factors

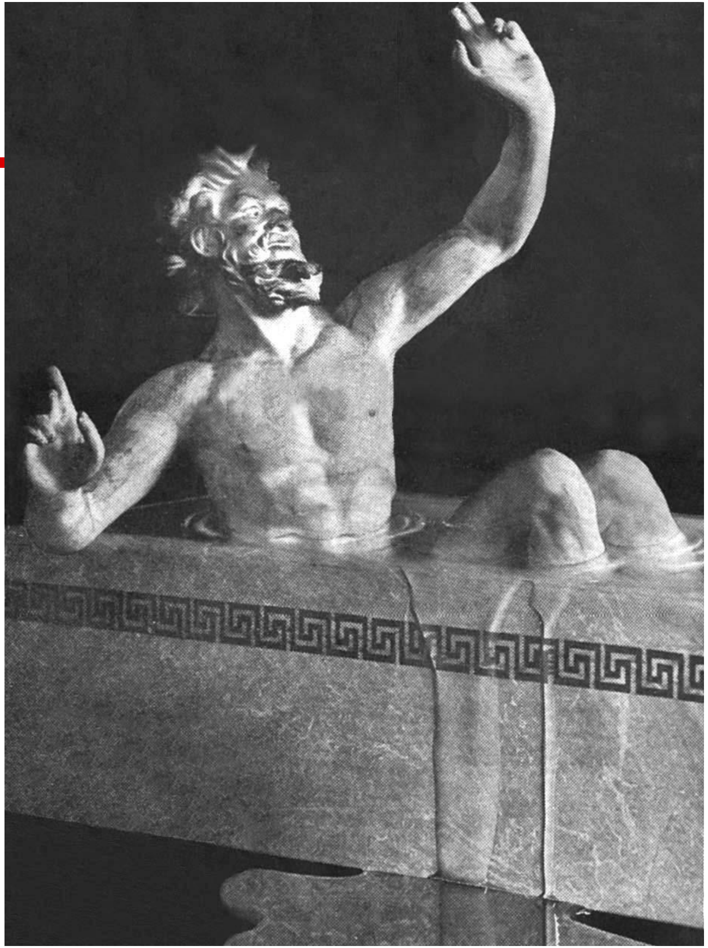


Problem

- Attached or Induced mass was first proposed by Friedrich Bessel in 1828
- Vertical oscillations increase the attached mass that affects the bubble
- Phenomenon of attached mass creates a changing effective gravitational potential

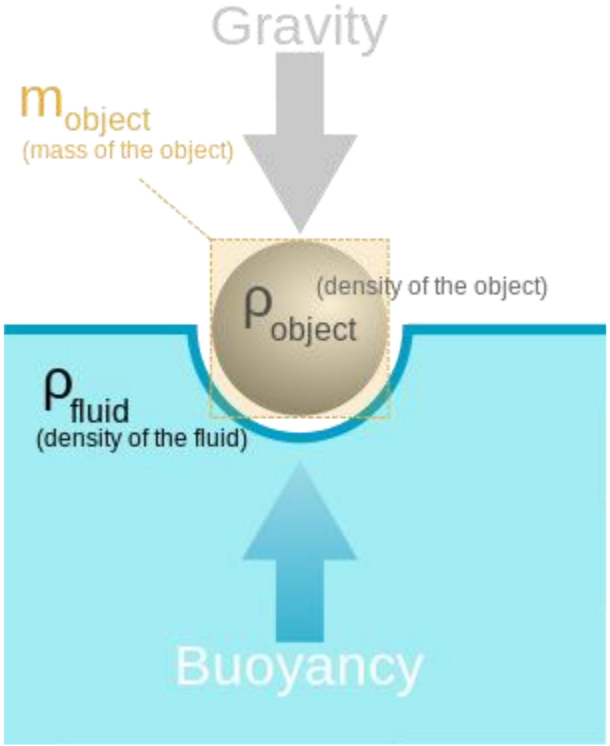
Model

- Concepts
  - Archimedes’ principle
    - buoyancy
  - Laplace pressure
    - pressure difference between bubble inside and outside
  - Friction
    - drag on bubble’s motion
  - Attached mass



Archimedes’ Principle

- The buoyant force arises from a mass displacing fluid
- Depends on the density and volume displaced
- $F = \rho V g$
- Archimedes postulated that the difference in pressure between an upper and lower face of an object caused this effect

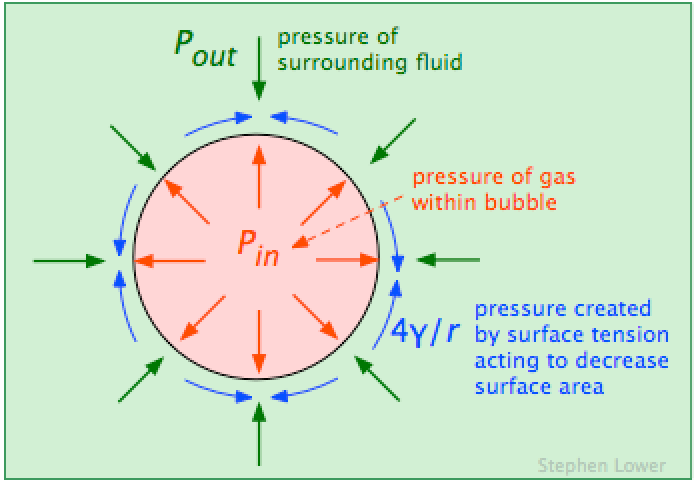


Laplace Pressure

- Pressure difference between two sides of a curved surface
- Arises from surface tension γ of the interface
- For spheres, Young-Laplace equation reduces to the second equation
- Smaller droplets have non-negligible extra pressure
- Commonly used for air bubbles in water or oil bubbles in water

$$\Delta P = P_{inside} - P_{outside} = \gamma\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

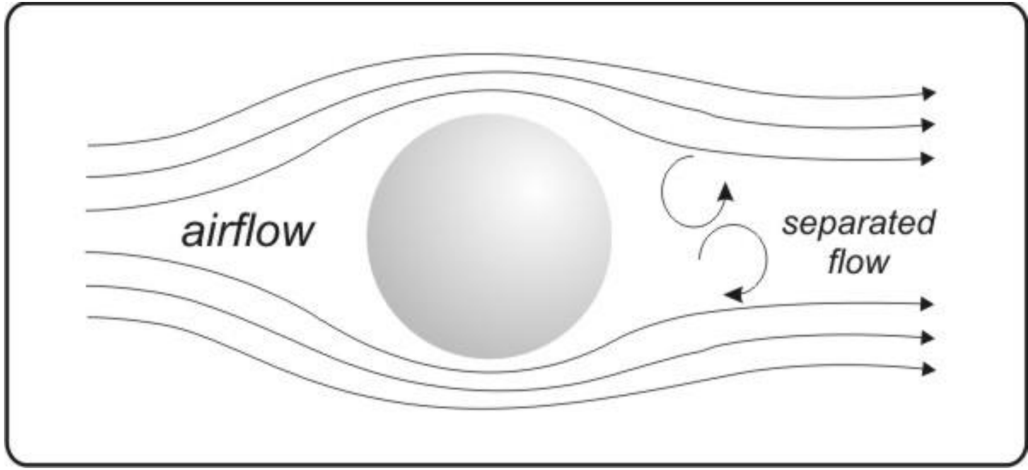
$$\text{For sphere: } \Delta P = \frac{2\gamma}{R}$$



Stephen Lower

Friction

- Also known as drag or fluid resistance
- Described mathematically by the drag equation
- Depends on the velocity of the object and fluid properties
- Drag coefficient C is dimensionless and describes drag in fluids



$$F_D = \frac{1}{2}\rho v^2 CA$$

Attached Mass

- Objects moving in fluids must accelerate fluid around them which increases inertia of system
- Various changes in velocity affect the Kinetic energy of fluid
- Object must do work to increase fluid’s kinetic energy
- The attached mass determines the work done to change the kinetic energy

$$T = \frac{\rho}{2}\int_V \left(u_\mathrm{x}^2 + u_\mathrm{y}^2 + u_\mathrm{z}^2\right) dV$$

Attached Mass

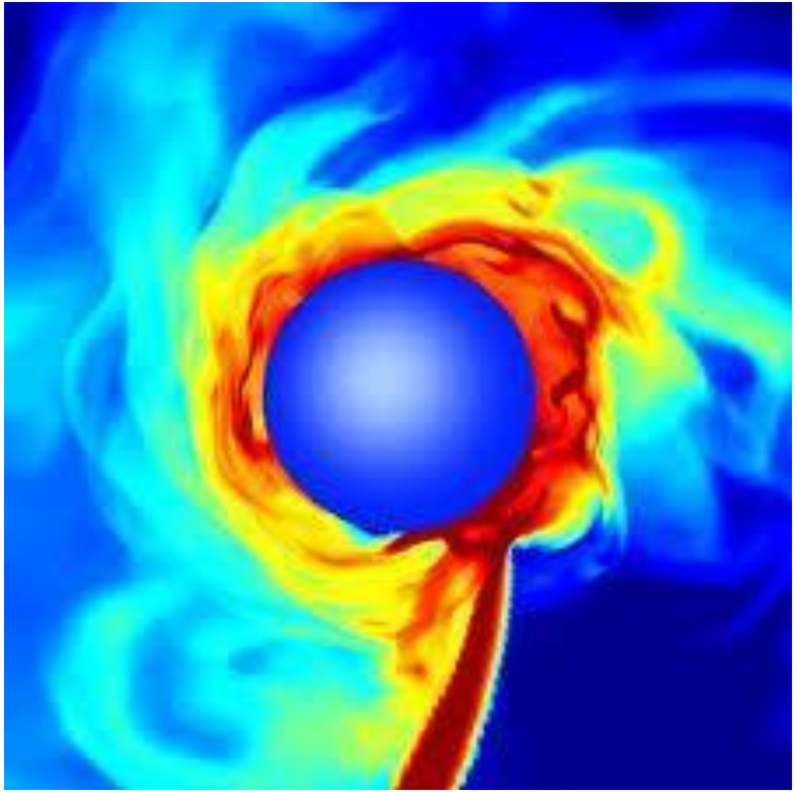
- Introduce I for the integral on the previous slide which represent how the volume of the sphere changes with velocity
- Attached mass is equal to I times density. Therefore, attached mass for spherical objects is one half the mass of the displaced fluid.

$$I = \frac{2}{3}\pi R^3 \quad T = \rho \frac{I}{2} U^2$$

$$m_{att} = I\rho_{fluid} = \frac{\frac{2}{3}\pi R^3 \, m_{fluid}}{\frac{4}{3}\pi R^3} = \frac{1}{2} m_{fluid}$$

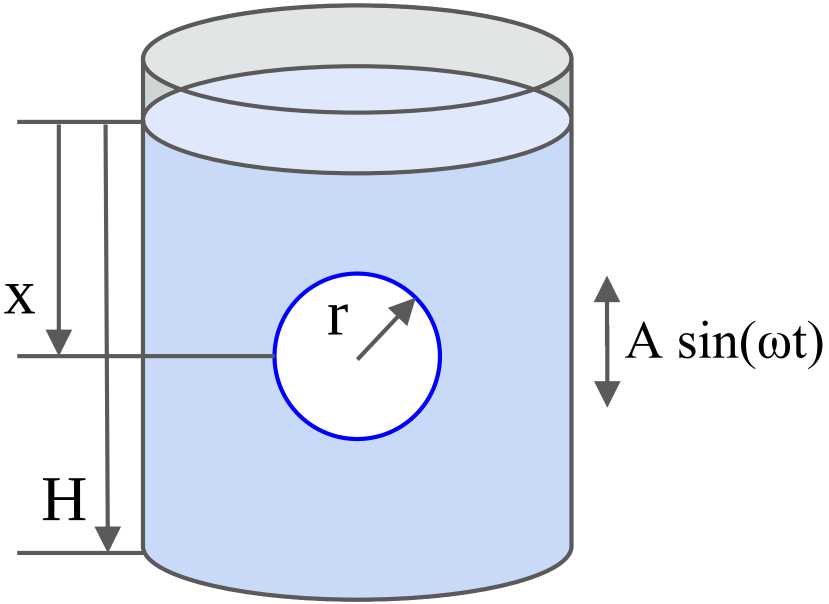
Induced Mass Concept

- Attached mass can be modeled as though bubble is dragging fluid with it
- In reality all of the fluid in system is accelerating



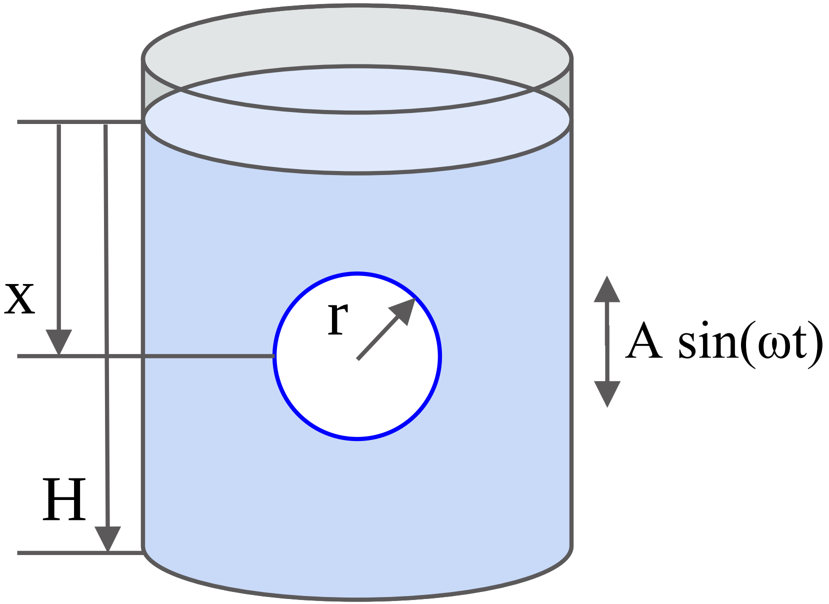
Model

- Assumptions
    - Spherical bubbles
    - Incompressible liquid
        - $\nabla \cdot \vec{V} = 0$
    - Container is open on top
    - Bubble volume changes are insignificant (quasistatic)
    - Ideal pressure conditions
        - too much = no oscillations
        - too little = cavitation



Model

- Parameters
  - Total water depth - H
  - Bubble depth - x
  - Bubble radius - r
  - Oscillation amplitude - A
  - Oscillation frequency - ω
  - Time duration - t



Bubble Volume

Assume the bubble to be isothermal (Surface Area Dominates Volume):

- The Ideal Gas Law implies that:

$$P(t)V(t) = P_0V_0$$

- Fluid oscillations implies the pressure is:

$$P(t) = P_0 + \rho x(g + A\omega^2 \sin \omega t)$$

- The final result for the volume of the bubble is:

$$V(t) = \frac{P_0V_0}{P_0 + \rho x(g + A\omega^2 \sin \omega t)}$$

Model

- Combining these varying functions into a general equation results in this differential equation of motion for the bubble:

$$(m + m_{\text{att}})\ddot{x} + \dot{m}_{\text{att}}\dot{x} = -F(\dot{x}) + (m - \rho V(t))(A\omega^2 \sin \omega t + g)$$

- Bubble's mass – m
- Attached mass – m_{att}
- Drag force – $F(\dot{x})$
- $F(\dot{x}) = 4\rho R^2 \psi(Re)\dot{x}^2 \operatorname{sgn} \dot{x}$
- Bubble's Volume – $V(t)$
- Buoyancy term – ρV_b
- Oscillating fluid term – $A\omega^2 \sin \omega t$

Model

- This is the equation used in our computational models:

$$(m + m_{\mathrm{att}})\ddot{x} + \dot{m}_{\mathrm{att}}\dot{x} = -F(\dot{x}) + (m - \rho V(t))(A\omega^2 \sin \omega t + g)$$

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Model - Separation of Variables

- Method of Separation of Variables
- Harmonics of these types of oscillations imply that one can assume that the solutions are of the form: $x(t,\tau) = X(t) + \Psi(\tau)$
- $X(t)$ is the ‘slow’ solution (represents the trajectory of the bubble)
- $\Psi(\tau)$ is the ‘fast’ solution (represents the rapid oscillation of the bubble)

Time Average Position of the Bubble

Average Position of Bubble: $\langle\, x(t,\tau)\,\rangle = \langle\, X(t)\,\rangle + \langle\, \Psi(\tau)\,\rangle$

Slow Fast

- Since $\Psi(\tau)$ is periodic its average is zero
- Therefore the average position of the bubble is described by the changes that take place slowly in time

Derivation

Since the slow equation has terms that depend on the fast equation, the fast equation must be solved first.

$$(m + m_{att})\ddot{\Psi} = -4\rho R_0^2 \psi_\infty \dot{\Psi}^2 sgn\left(\dot{\Psi}\right) - \langle \dot{\Psi}^2 \operatorname{sgn} \dot{\Psi} \rangle + \left(m - \rho V(t)\right) A\omega^2 \sin \omega t$$

Derivation

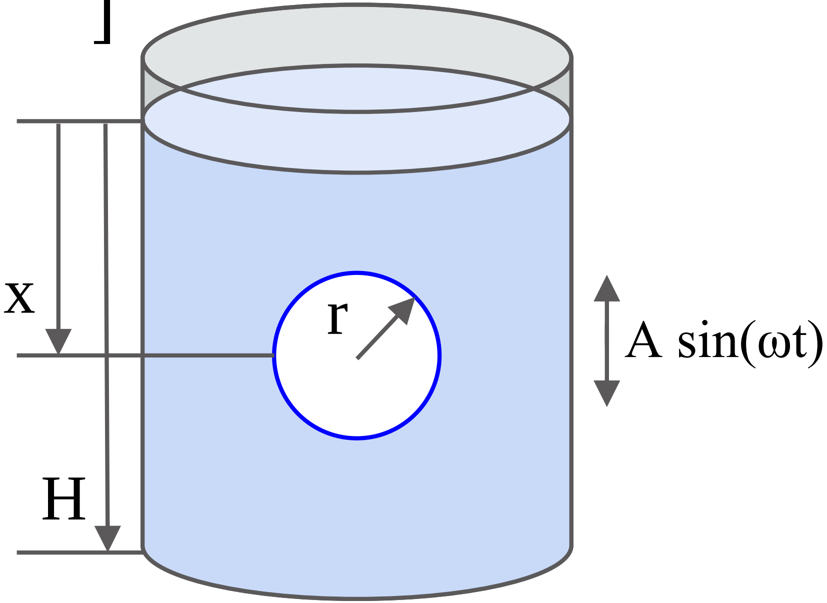
Solving the fast equation in an approximate manner lead us to a final slow equation of the following form

$$m_{att}\ddot{X} + \frac{16}{\pi}\rho R_0^2 \psi_\infty \dot{X} B\omega = \gamma\omega^2 \frac{X\rho V(t) g}{2H}\left(1 - \frac{2\theta\left(\frac{A^2}{R_0^2}\right)}{6\left(1+\sqrt{1+\theta\frac{A^2}{R_0^2}}+\theta\frac{A^2}{R_0^2}\right)}\right) - \rho V(t) g$$

$$\theta = \frac{16^2\,\psi_\infty^2}{\pi^4\,X^4}$$

Velocity of Bubble

- Acceleration of bubble is relatively small $\dot{x} \approx v\left[\frac{x}{x_0} - 1\right]$
- Results in 3 cases
  - Bubble sinks $x > x_0$
  - Bubble remains motionless $x = x_0$
  - Bubble floats $x < x_0$



Computational Model

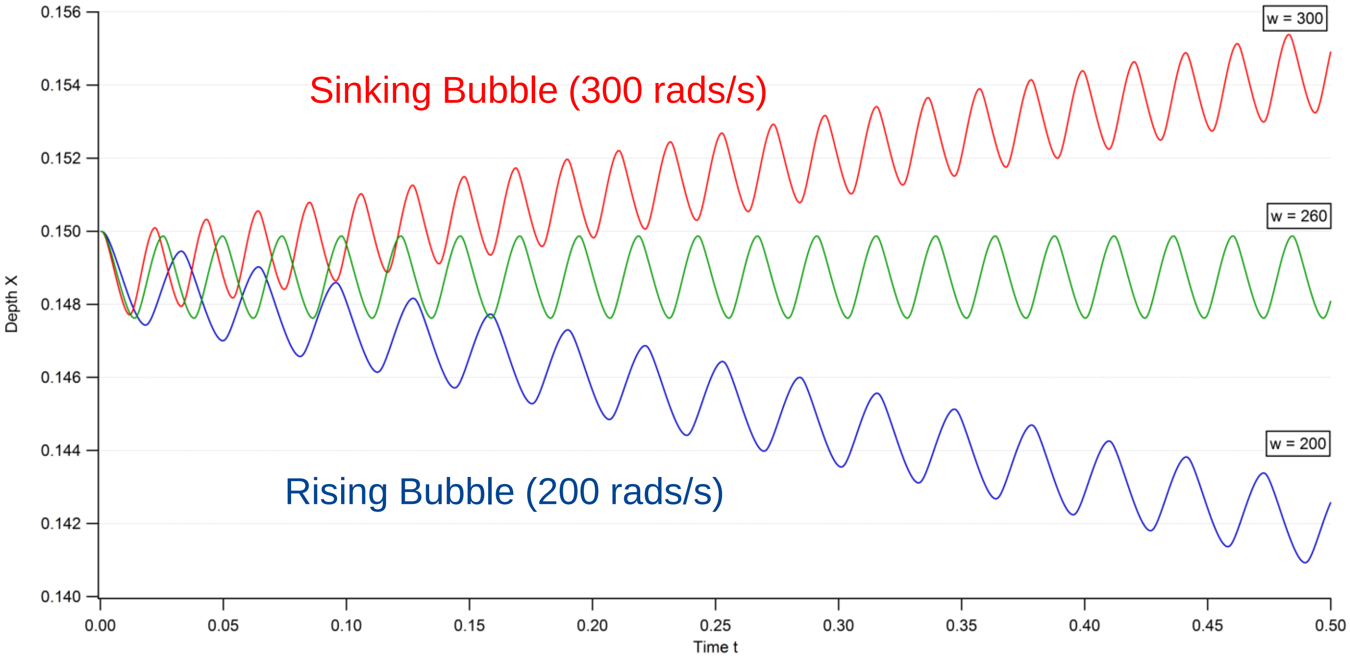
$$(m + m_{\mathrm{att}})\ddot{x} + \dot{m}_{\mathrm{att}}\dot{x} = -F(\dot{x}) + (m - \rho V(t))(A\omega^2 \sin \omega t + g)$$

- For our computational model we used the governing equation without any approximations
- We used a modified Verlet integration method using Matlab
- The dependant factors we studied were our frequency and initial position

- Bubble’s mass – m
- Attached mass – m_{att}
- Drag force – $F(\dot{x})$
- Bubble’s Volume – $V(t)$
- Buoyancy term – ρV_b
- Oscillating fluid term – $A\omega^2 \sin \omega t$

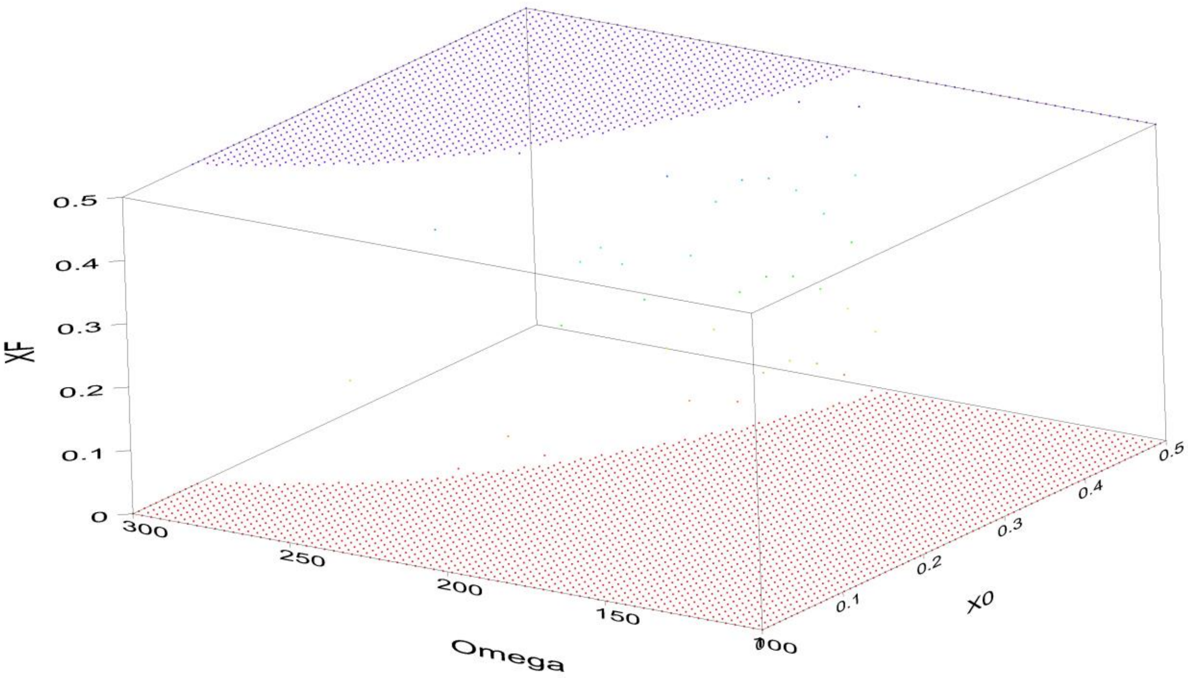
Velocity of Bubble

- The bubble's position will also change depending on the frequency of induced oscillations
- Higher frequency causes the bubble to sink
- Lower frequency causes the bubble to rise



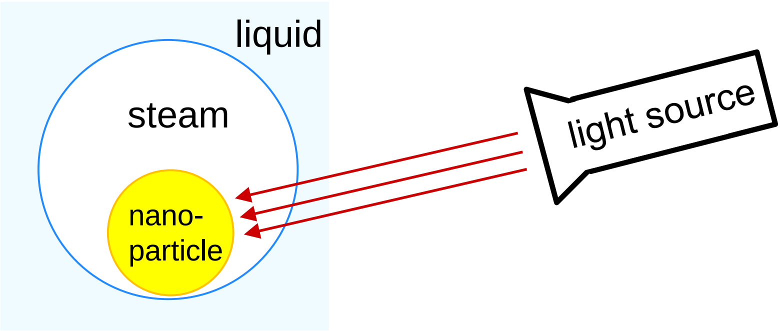
Bifurcation Diagram

- Given large enough time duration, the unstable nature of the solution creates two regions for solutions
- Blue region represents conditions where bubbles sink
- Red region represents conditions where bubbles rise



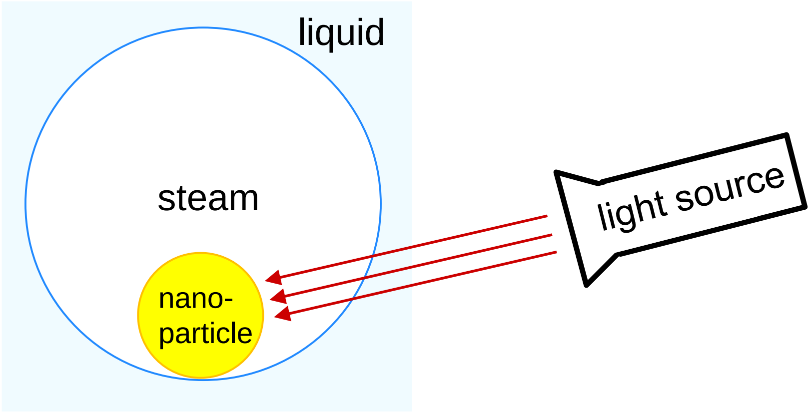
Plasmonic Nano-Particles

- Introduction of nano-particles made of gold causes interesting effects
- The particles are plasmonic meaning their electron density couples with electromagnetic fields
- Plasmonic nano-particles will oscillate at the frequency of incident light within a certain frequency range



Plasmonic Nano-Particles

- The oscillating particles heat the fluid and cause steam bubbles to be generated
- Over time the temperature increase by the light absorption of the particles increases the volume of the bubble



Adiabatic Solution

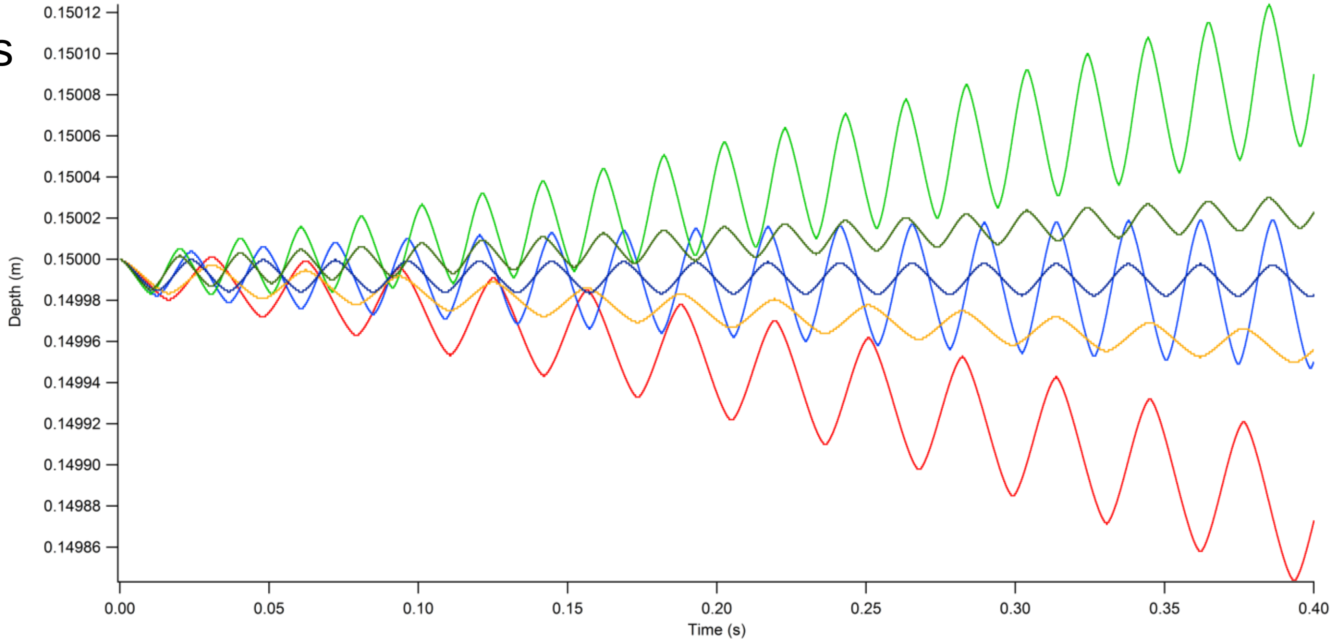
- The ideal gas law is applied to the system to derive the following term for the volume:

$$V(t) = \frac{P_0 V_0 \left(1 + \frac{I_{inc}\sigma t}{T_0 4\pi k R_{np}}\right)}{P_0 + \rho g x + \rho x A \omega^2 \sin(\omega t)}$$

- This term is added into our differential equation and computationally modeled.

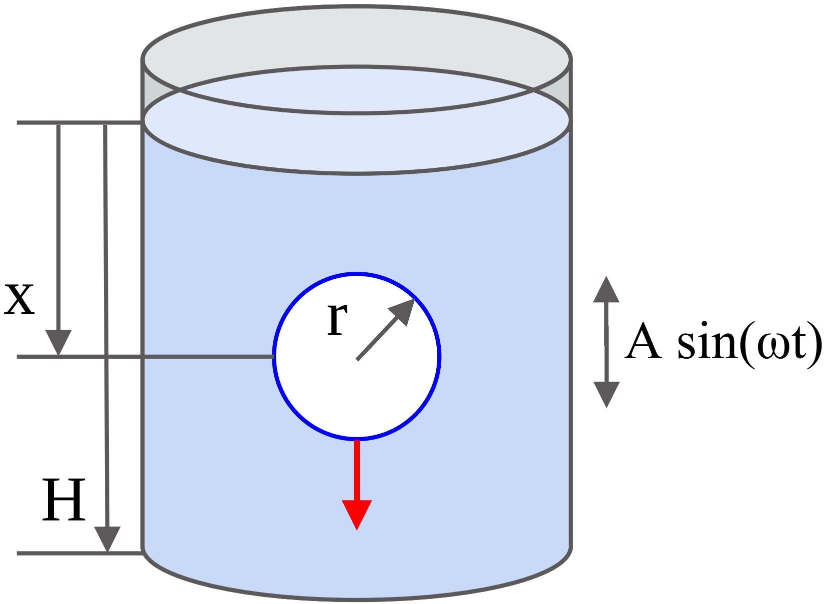
Solution

- As the nano-particles are heated the bubbles grow in volume
- Increase in bubble volume increases attached mass
- Result is increase in rate of bubble movement, both slow and fast motion



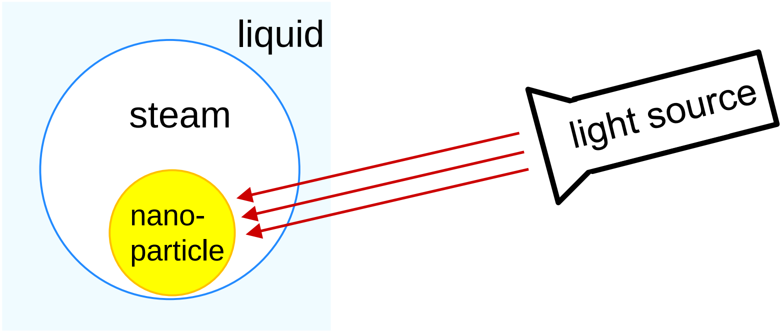
Conclusion

- Bubbles in vibrating fluids will sink given certain circumstances
    - Dependent factors
        - Bubble depth
        - Vibration frequency
        - Bubble volume
- Cause
    - Gravity’s effect on attached mass overcomes buoyancy force



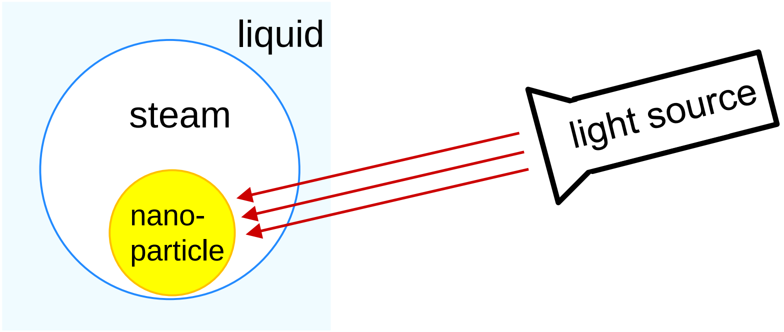
Conclusion

- Addition of nano-particles can cause the formation of bubbles and affect their movement
- Nano-particles will increase movement of bubbles when exposed to incident light
- Nano-particles convert solar energy to steam at high efficiency (>80%)



Potential Applications

- Shining light on water containing nano-particles can be used to create abnormally high temperature steam
- Can be used as an inexpensive solar autoclave for sterilization



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