# VIBRATING BASE PENDULUM

Math 485 Project team

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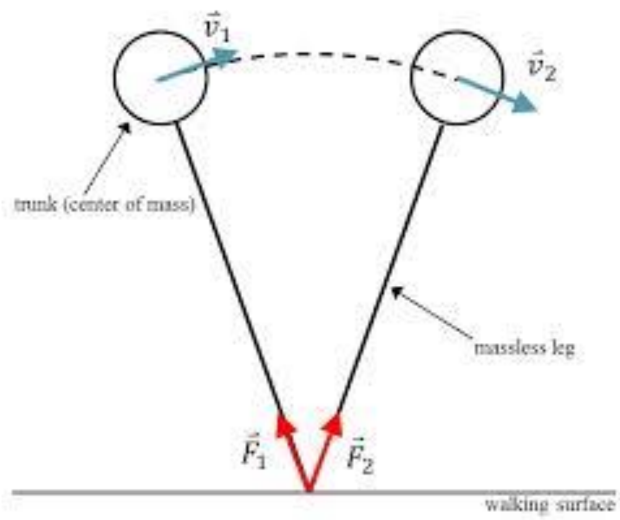
# Agenda

- Midterm Recap
- Equation of Motion & Energy Modeling
- Effective Potential
- Stability Analysis
- Experiment
- Error Analysis

# Background

Inverted Vibrating Pendulum




Application

# Recap--Vertical Angle

- Equation of Motion:

$$K = \frac{1}{2}m(\dot{\theta}^2 l^2 + {d_0}^2\omega^2\sin^2(\omega t) - 2\theta\dot{l}\ \ (\sin\theta)d_0\omega\sin(\omega t)\ )$$

$$\mathrm{U} = mg(l\cos\theta + d_0\sin(\omega t))$$

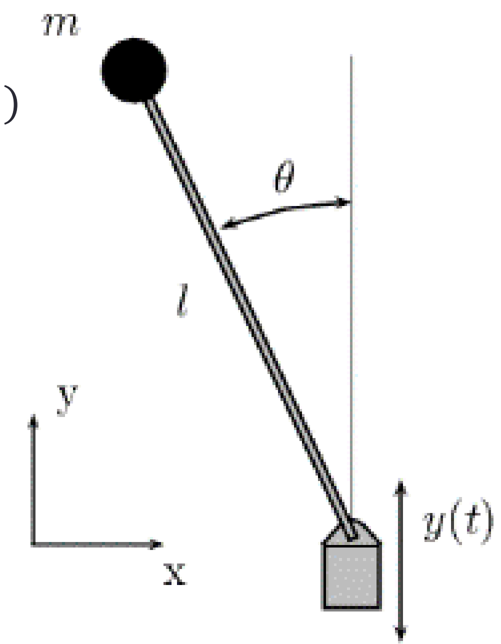
- Lagrangian:

$$L = K - U$$

$$L = \frac{1}{2}l\dot{\theta}^2 + d_0\omega\dot{\theta}sin\theta\, sin(\omega t) - gcos\theta + \frac{1}{2l}\omega^2$$

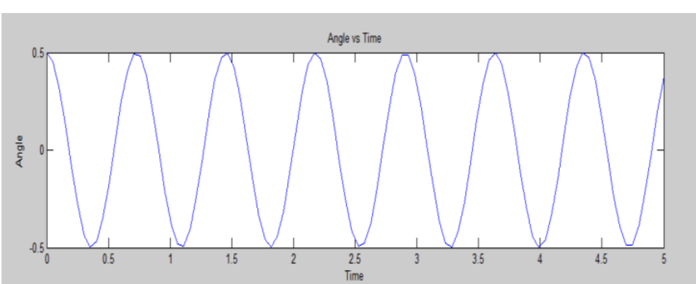
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \text{ (Euler-Lagrange Equation)}$$

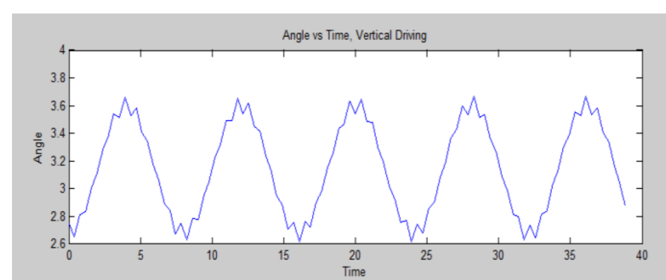
$$\ddot{\theta} + \left[\frac{d_0\omega^2}{l}cos(\omega t)\ - \frac{g}{l}\right]sin\theta = 0$$




- Separate into "fast" and "slow" motion

$$\theta(t) = X(t) + \xi(t)$$



# Recap—Vertical Angle

- Averaging

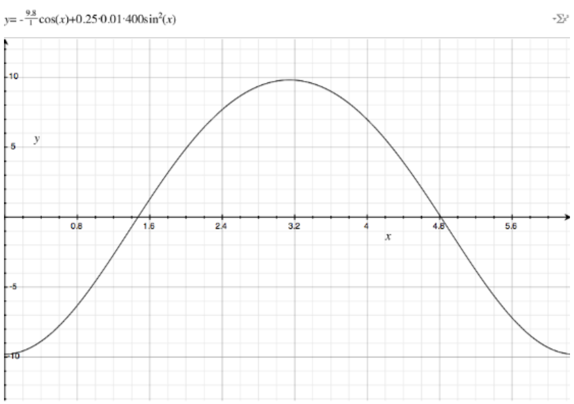
$$\xi = -\frac{d_0 \omega^2}{l} \sin(X) \iint \cos(\omega t)\, dt^2 = \frac{d_0}{l} \sin(X) \cos(\omega t)$$

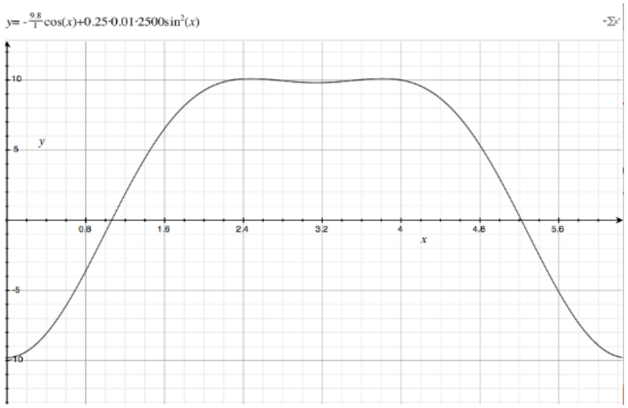
$$\ddot{x} = -\frac{d}{d\theta}\left(-\frac{g}{l}\cos(\theta) + \frac{1}{4}\frac{{d_0}^2 \omega^2}{l^2}\sin^2(\theta)\right)$$

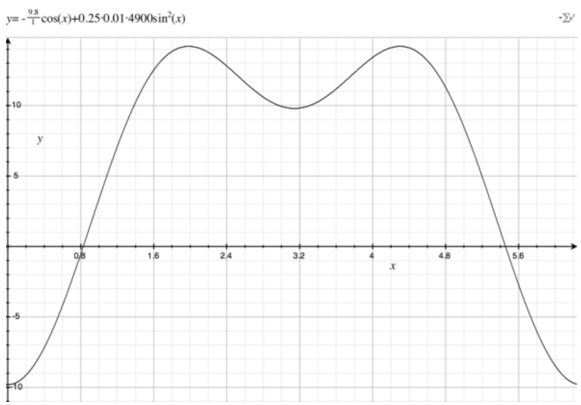
- Effective Potential

$$U_{eff} = -\frac{g}{l}\cos(\theta) + \frac{1}{4}\frac{{d_0}^2 \omega^2}{l^2}\sin^2(\theta)$$

- Stability Analysis



# Objectives

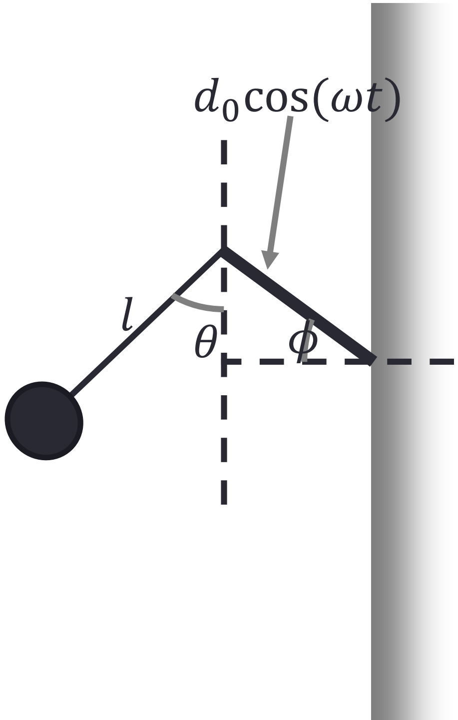
Lagrangian → Effective Potential → Stability Analysis

Horizontal Angle → Arbitrary Angle

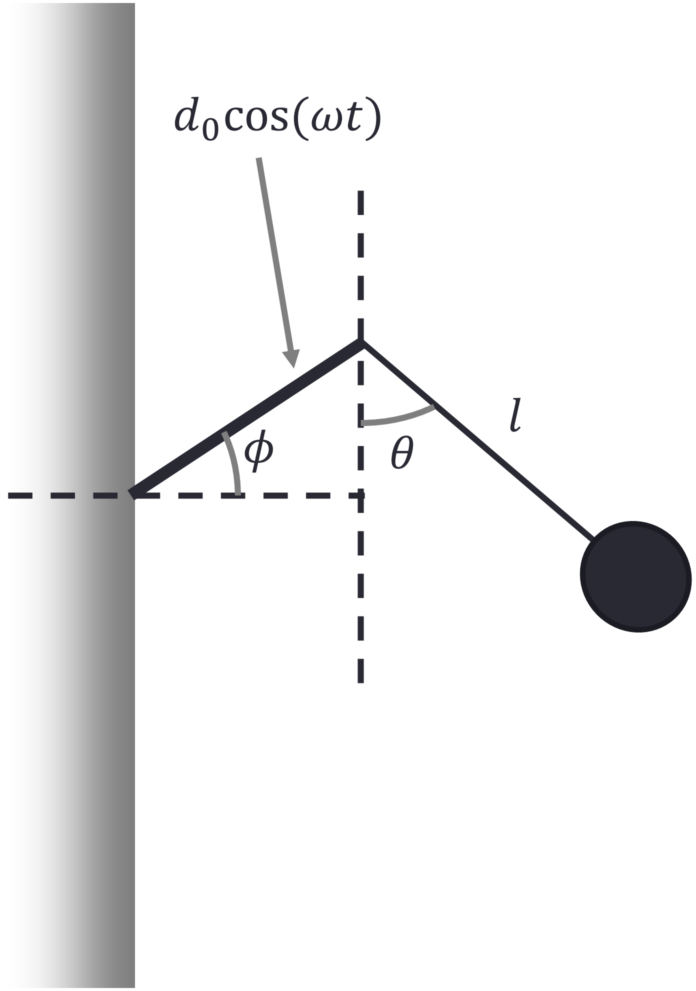
Experiment

# Variables

- $d_0 = amplitude\ of\ base\ oscillations$
- $\omega = angular\ frequency\ of\ base\ oscillations$
- $l = length\ of\ pendulum$
- $\theta = counterclockwise\ angular\ displacement\ of\ pendulum$
- $\phi = counterclockwise\ angle\ of\ base$
- $g = gravitational\ constant\ (9.81\ m/s^2)$
- $K = kinetic\ energy$
- $U = potential\ energy$



# Arbitrary Angle of Base




X & Y Coordinates:

- $x = l sin(\theta) + d_0 \cos(\omega t) \cos\phi$
- $y = l - l cos(\theta) + d_0 \cos(\omega t)\, sin\phi$

Velocities:

- $v_x = l\dot{\theta} \cos(\theta) - d_0\omega \sin(\omega t) \cos\phi$
- $v_y = l\dot{\theta}\sin(\theta) - d_0\omega \sin(\omega t)\, sin\phi$

# Lagrangian for Arbitrary Angle

- Lagrangian for any physical system is defined as Kinetic Energy minus Potential Energy

$$L = K - U$$

- Kinetic Energy:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2)$$

$$K = \frac{1}{2}ml^2\dot{\theta}^2 - md_0\omega l\dot{\theta}\cos(\theta - \phi)\sin(\omega t)$$

- Potential Energy:

$$U = mgh$$

$$U = mgl - mgl\cos\theta + gd_0\cos(\omega t)\sin\phi$$

- Lagrangian:
- $L = \frac{1}{2}ml^2\dot{\theta}^2 - md_0\omega l\dot{\theta}\cos(\theta - \phi)\sin(\omega t) + mgl\cos\theta - gd_0\cos(\omega t)\sin\phi$
- $L = \frac{1}{2}l\dot{\theta}^2 + d_0\omega^2\sin(\theta - \phi)\cos(\omega t) + g\cos\theta$

# Effective Potential Derivation

- Use Euler-Lagrange Equation to write Equation of Motion

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

- Separate variables into rapid oscillations due to vibrating base and slow motion of pendulum

$$\theta = X + \xi$$

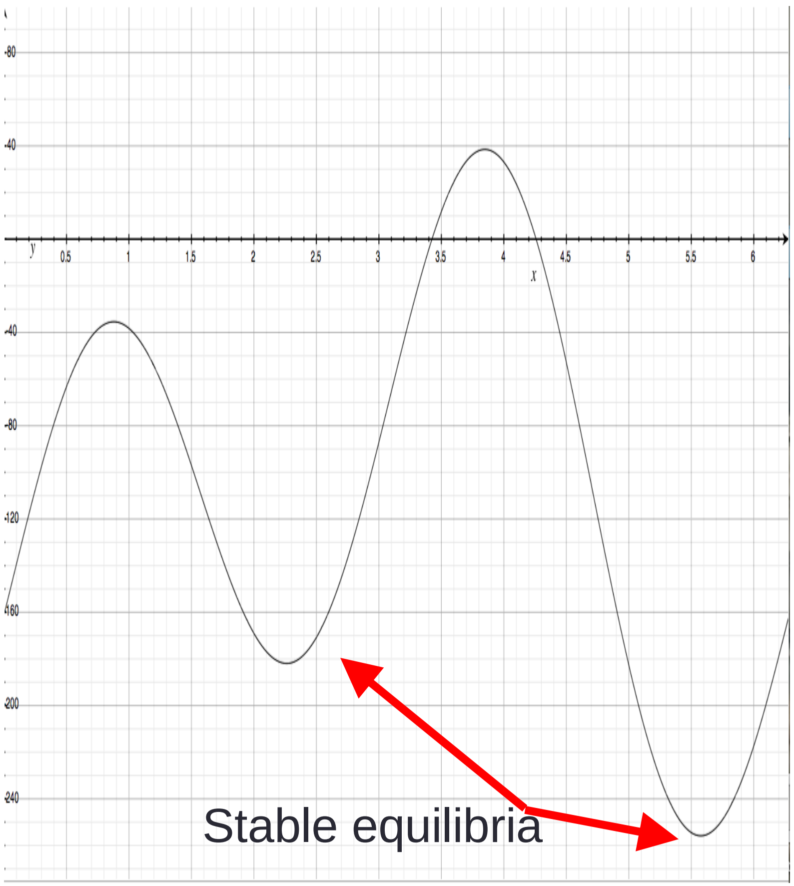
- Final differential equation can be written as a total derivative in position, which corresponds to the effective potential energy of the system

$$\ddot{X} = -\frac{\partial}{\partial X}\left(-\frac{g}{l}\cos\theta - \frac{d_0^2\omega^2}{4l^2}\sin^2(\theta - \phi)\right)$$

- General equation of motion relates position and potential energy

$$\ddot{x} = -\frac{\partial U(x)}{\partial x}$$

# Effective Potential




Effective potential of a pendulum with base angle of 45°,

- Comparing equation of motion to general form suggests concept of “effective potential”

$$U_{eff} = -\frac{g}{l}\cos\theta - \frac{d_o^2\omega^2}{4l^2}\sin^2(\theta - \phi)$$

- Separation of variables treats motion of the pendulum as one smooth motion with periodic perturbations
- Averaging technique smooths out rapid oscillations by averaging over the period of the rapid motion, like a strobe light, creating an idealized model
- “Effective potential” is the hypothetical potential energy of the idealized model

# Stability Analysis

- Stability occurs at local minima of potential energy, including effective potential
- Stability positions appear for frequencies above a minimum frequency

$$\omega > \frac{\sqrt{2gl}}{d_0}$$

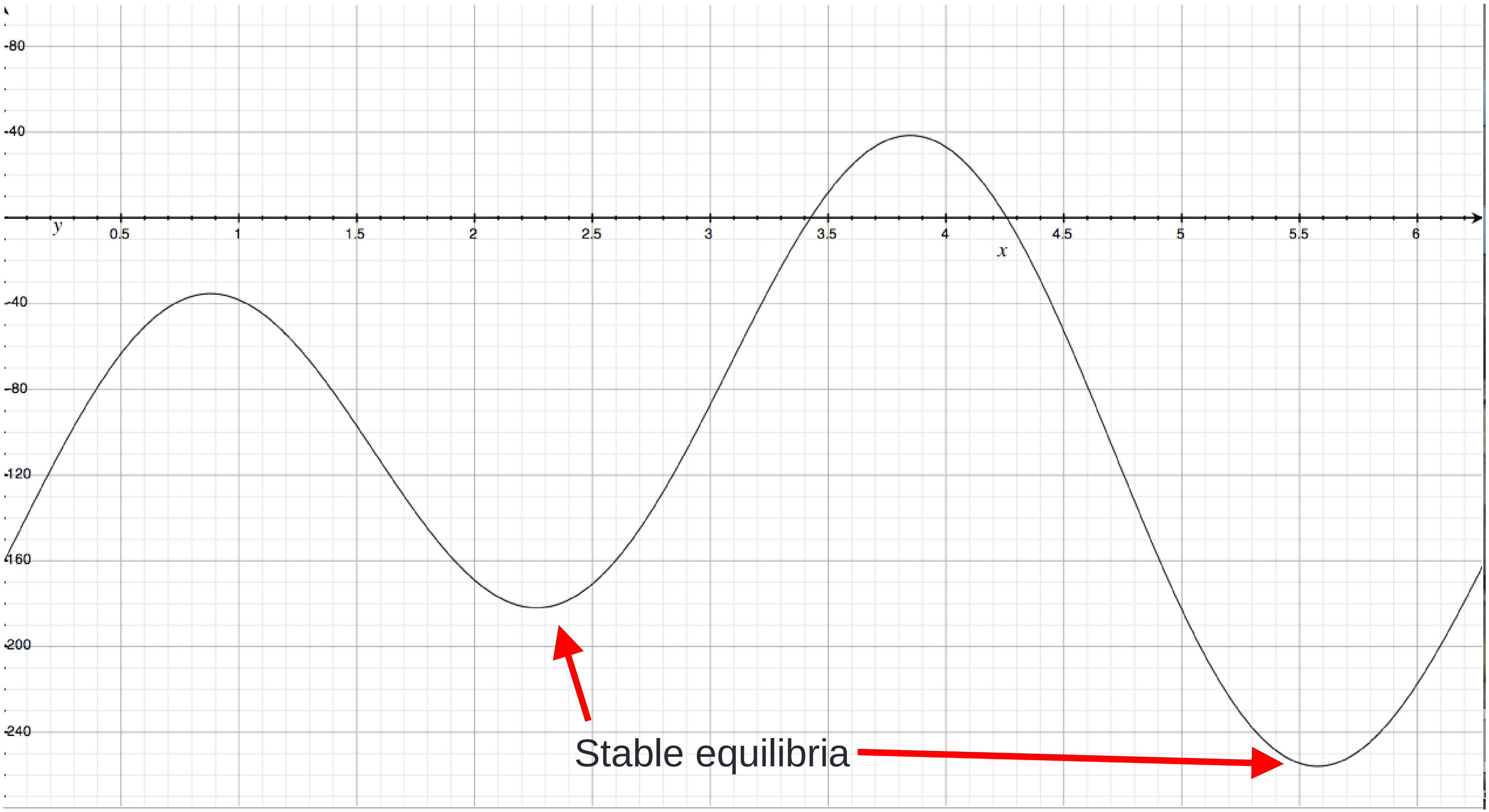
- For Horizontal this occurs at an angle:

$$\theta_s = \cos^{-1}\left(\frac{2gl}{d_0^2\omega^2}\right)$$

- For an arbitrary angle of the base of 45°, theoretical stable angle is129°, or 39° above the horizontal

$$\omega = 275.62\frac{rad}{sec} \qquad l = 0.187\,m \qquad d_0 = 0.02\,m$$

# Stability Analysis



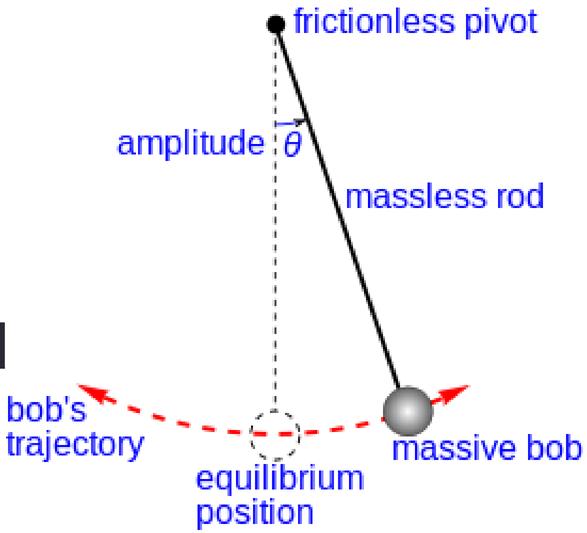

$$\omega = 275.62\,\frac{rad}{sec} \qquad l = 0.187\ m \qquad d_0 = 0.02\ m \qquad \phi = 45°$$

# Physical Pendulum

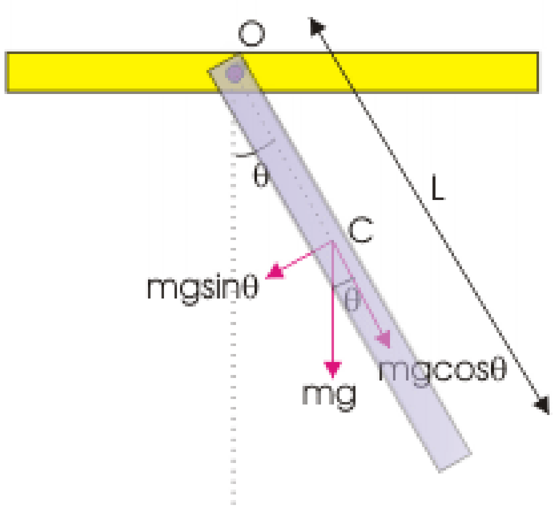
- Theoretical model used "simple pendulum" where all mass is concentrated at single point at end of rod
- "Physical pendulum" is realistic model
- Must incorporate center of mass and moment of inertia into calculations

$$U_{eff} = -\frac{6}{7}\frac{g}{l}\cos\theta - \frac{36}{49}\frac{d_o^2\omega^2}{4l^2}\sin^2(\theta - \phi)$$

- Shifts stability points to more realistic locations



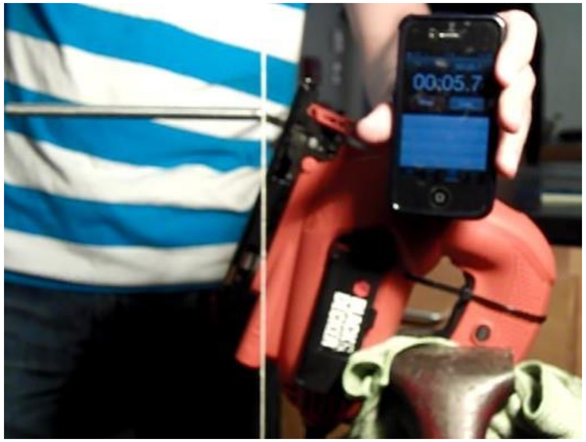
Simple pendulum model



Physical pendulum model

# Experimental

- Overview
  - Use Cannon High Speed Camera to observe the pendulum's motion
  - On screen measurement




| Raw Data Table | Measurement |
|---|---|
| Length of Pendulum (m) | .187 |
| Diameter of Pendulum (m) | 0.009525 |
| Amplitude (m) | 0.020 |
| Minimum | 0.010 |
| Maximum | 0.030 |
| Frequency (rad/s) | 275.62 |
| Angle of Base | 51° |
| Moment of Inertia | $I = \frac{1}{3}ml^2$ |

# Experimental

- Results

| Measurement | Theoretical | Experimental | Corrected Theoretical |
|---|---|---|---|
| **Vertical Stability Angle** | 180° | 180° | 180° |
| **Critical Angle** | 97° | 113° | 98° |
| **Horizontal Stability Angle** | 83° | 76° | 82° |
| **Arbitrary Stability Angle** | 136° | 109° | 135° |

$$\omega = 275.62 \frac{rad}{sec} \qquad l = 0.187\ m \qquad d_0 = 0.020\ m \qquad \phi = 51°$$

# Error Analysis

## Error in Measurements

When taking measurements, there will be some error that depends on the accuracy of the instrument. This absolute error is called the “Least Count”.

Relevant Measurements:

| | Measured Value | Absolute Error | Percent Error |
|---|---|---|---|
| **Length of Pendulum ($l$)** | $l$ = 0.187 meters | $\delta l$ = 0.001 meters | .54% |
| **Amplitude of Base ($d_\circ$)** | $d_\circ$ = 0.020 meters | $\delta d_\circ$ = 0.001 meters | 5.0% |
| **Period for 60 Oscillations (T*)** | T* = 1.35 seconds | δT* = 0.05 seconds | 3.7% |

Note that Angular Frequency (ω) cannot be directly measured. Instead, the variable T* is introduced and defined as the period of time required for the pendulum’s base to make 60 oscillations.

$$\omega = 2\pi \frac{60}{T^*} = \frac{120\pi}{T^*}$$

# Error Analysis

## Error Propagation

For a function f(x,y) with absolute errors δx and δy, there is sure to be some propagation of absolute error δf. This error is given by the Variance Formula:

$$\delta f(x,y,z) = \sqrt{\left(\frac{\partial f}{\partial x}\delta x\right)^2 + \left(\frac{\partial f}{\partial y}\delta y\right)^2 + \left(\frac{\partial f}{\partial z}\delta z\right)^2}$$

## Relevant Equations For Pendulum

Critical Angles for Vertical Pendulum

$$\theta_c(l, d_0, \omega) = \pm\cos^{-1}\left(-\frac{glT^{*2}}{7200\pi^2 d_0^2}\right) = \pm 97°$$

Stability Angle for Horizontal Pendulum

$$\theta_c(l, d_0, \omega) = \cos^{-1}\left(\frac{glT^{*2}}{7200\pi^2 d_0^2}\right) = 83°$$

# Error Analysis

$$\theta_c(l, d_0, \omega) = \pm\cos^{-1}\left(\alpha\frac{lT^{*2}}{d_0^2}\right) = \pm 97°$$

## Error for Theoretical Critical Angles

The Variance Formula for $\theta(l, d_0, \omega)$:

$$\delta\theta(l, d_0, \omega) = \sqrt{\left(\frac{\partial f}{\partial l}\delta l\right)^2 + \left(\frac{\partial f}{\partial d_0}\delta d_0\right)^2 + \left(\frac{\partial f}{\partial \omega}\delta\omega\right)^2}$$

By letting $\alpha = -\frac{g}{7200\pi^2} = -1.38\ x\ 10^{-4}$ the following partial differential equations can be obtained:

$$\frac{\partial\theta}{\partial l} = -\frac{\alpha T^{*2}}{d_0^2\sqrt{1-\left(\frac{\alpha l T^{*2}}{d_0^2}\right)^2}} \qquad \frac{\partial\theta}{\partial T^*} = -\frac{2\alpha l T^{*}}{d_0^2\sqrt{1-\left(\frac{\alpha l T^{*2}}{d_0^2}\right)^2}} \qquad \frac{\partial\theta}{\partial d_0} = \frac{2\alpha l T^{*2}}{d_0^3\sqrt{1-\left(\frac{\alpha l T^{*2}}{d_0^2}\right)^2}}$$

**Notice that since each of these quantities will be squared, the sign doesn't matter. This is why the error analysis will be the same for the stability angle for the Horizontal Pendulum.

# Error Analysis

$$\delta\theta(l, d_0, \omega) = \sqrt{\left(\frac{\partial f}{\partial l}\delta l\right)^2 + \left(\frac{\partial f}{\partial d_0}\delta d_0\right)^2 + \left(\frac{\partial f}{\partial \omega}\delta \omega\right)^2}$$

If we plug in our measured values, we obtain:

$$\delta\theta(l, d_0, \omega) = \sqrt{(0.633 * 0.001)^2 + (0.176 * 0.05)^2 + (11.84 * .001)^2}$$

$$\delta\theta = .0148\, radians = .86°$$

Percent Error = $\frac{\delta\theta}{\theta}\ x\ 100$

This means:

Critical Angles for the Vertical Pendulum: $\theta_c$ = ± 97 ± 0.86° = ± 97 ± 0.89%

Stability Angle for the Horizontal Pendulum: $\theta_s$ = 83 ± 0.86 ° = 83 ± 1.04%

# Thank You

Questions?