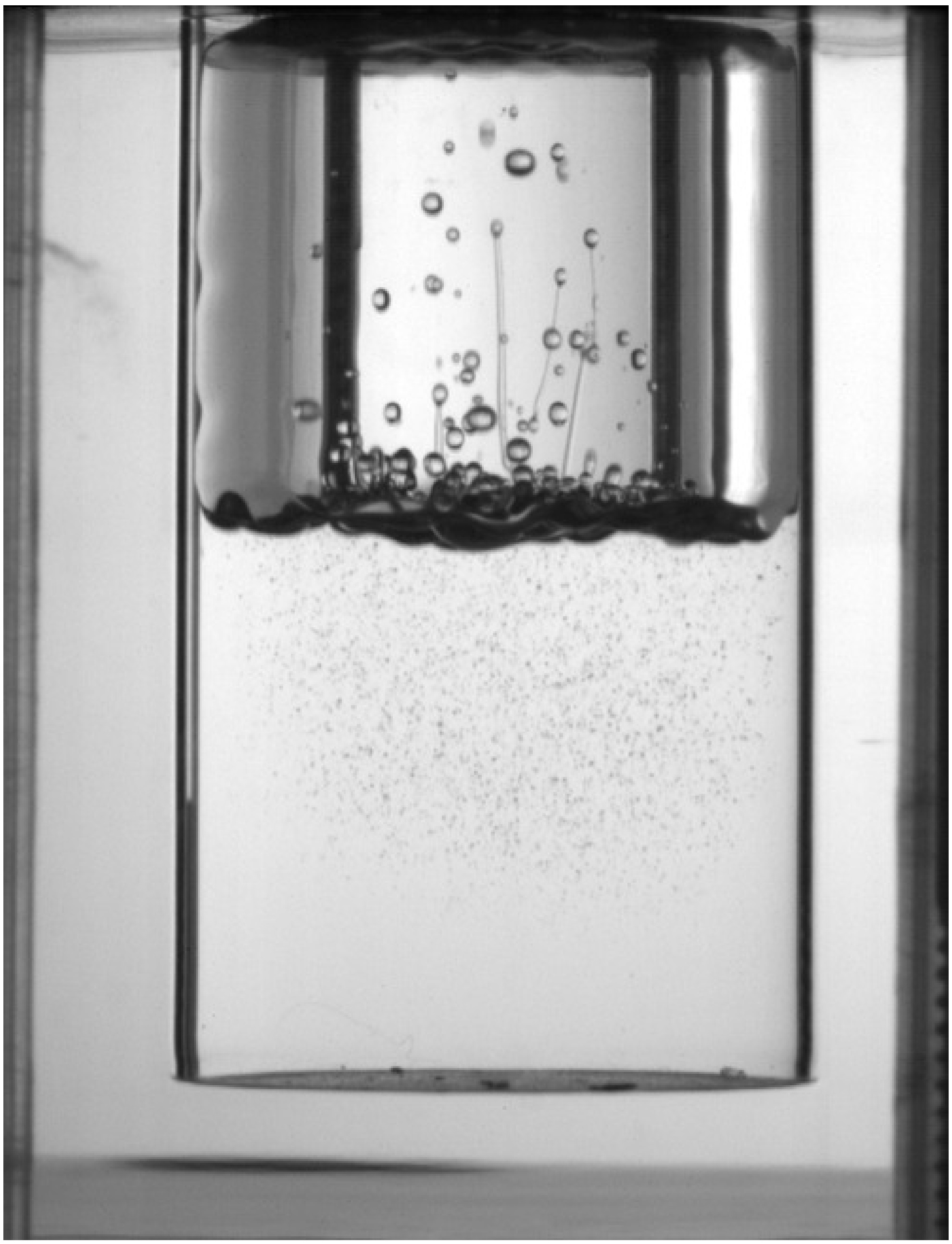
Sinking Bubbles in an Oscillating Liquid

James Wymer, Michael McIntire, Mohammad Ghallab, Jaggar Henzerling, Aaron Kilgallon

March 2014



Abstract

An interesting phenomena arises in fluid mechanics in which air bubbles will sink under certain circumstances. Several experiments have confirmed that when one oscillates a container parallel to gravity, or another accelerating force, bubbles in that fluid can sink rather than rise. We derive the equation of motion that describes the dynamics of bubbles in this system and use computational simulations to model their behavior. After our simulations, we determined a low initial position for the bubble and a high vibration frequency were necessary factors to causing the unusual sinking phenomenon. We then model the addition of nano-particles into the liquid. The adiabatic process generating the bubbles increases the steam bubble size when incident light is shined upon the suspended nano-particles. This was computationally modeled and shown to affect the motion of the bubbles in a dynamic way.

Contents

1 Introduction

The dynamics of bubbles in a liquid have applications in everything from aerospace to medical sanitation. One might assume their movement is predictable, but unusual effects occur when one vibrates the container the bubbles are in. Under specific frequencies of vibration, bubbles will sink and can coalesce in the bottom of a tank. A vertically oscillating container is a common occurrence when considering fuel tanks for rockets or other transportation technologies and these sinking bubbles can easily interfere with poorly placed fuel sensors. By conducting a thorough analysis we determined there is a narrow range of frequencies which can be used to control the bubbles vertical motion. Additional factors which affect the bubble's dynamics include the density and pressure of the fluid and the bubble's initial position. By monitoring or controlling these factors, one can use the dynamics to one's benefit or at least mitigate any negative effects they may have.

2 Model Parameters

2.1 Isothermal Bubbles

One generalized assumption made to fit the model was that the bubble size is very small. By assuming that the bubble size is small, it follows that the surface area of the bubble affects the temperature much more than the volume. Due to this, the bubble will diffuse its internal heat very quickly into the vibrating fluid. We then assume that the heat is fully diffused into the surrounding liquid, and that the temperature of the bubble is approximitely constant. We additionally assume that the shape of the bubble is spherical since at such a small volume any effects from an irregular shape would be insignificant.

By using the Ideal Gas Law, we find the pressure and volume of the bubble to be:

$$P(t)V(t) = P(0)V(0) \tag{1}$$

We then take into account the vibration of the liquid, which leads to oscillating pressure, as follows:

$$P_v(t) = A\sin(\omega t) \tag{2}$$

The total pressure on the bubble is therefore given by:

$$P(t) = P(0) + \rho g x + \rho x A\omega^2 \sin(\omega t) \tag{3}$$

Where $P(0)$ is the pressure outside the liquid, and $\rho g x$ is the increased pressure as a function of depth. Application of equations 1 and 3 imply that the volume of the bubble, as a function of time, is:

$$V(t) = \frac{P(0)V(0)}{P(0) + \rho x(g + A\omega^2 \sin(\omega t))} \tag{4}$$

This can be Taylor expanded into the following equation, with the assumption that the Laplace pressure greatly overwhelms the other forces acting on the bubble.

$$V(t) = V(0)[1 - \frac{\rho x g}{P(0)}(1 + \frac{A\omega^2}{g} sin(\omega t))] \tag{5}$$

2.2 Buoyancy Force

Archimedes' Principle states that the buoyant force on an object submerged in water is given by:

$$|\vec{F}| = \rho V g \tag{6}$$

which states that the upward force on a submerged object is equal to the weight of the fluid displaced by the object. This arises from a difference in pressure between the upper and lower ends of a body in fluids. The lower end will have a higher pressure, and will accelerate the body upwards.

2.3 Drag Force

The drag force on an object moving through a liquid is given by:

$$|\vec{F_D}| = \frac{1}{2}\rho v^2 C_D A \tag{7}$$

Here, C_D is the experimentally determined drag coefficient of the particular object in the liquid, A is the cross-sectional area, and v is the velocity of the object in the fluid.

This can be written more generally in the form:

$$F(\dot{x}) = |\vec{F_D}| sgn(\dot{x}) \tag{8}$$

Where $F(\dot{x})$ defines the drag force in terms of the magnitude of the drag force, and the $sgn(\dot{x})$ function, which forces $F(\dot{x})$ to counteract the direction of motion.

The magnitude of the drag force is given by:

$$|\vec{F_D}| = 4\rho R^2 \Psi(Re) x'^2 \tag{9}$$

R is the radius of the bubble, and $\Psi(Re)$ is the coefficient of resistance, with Re being the Reynolds number as defined by:

$$Re = \frac{2\rho RV}{\mu} \tag{10}$$

where μ is the viscosity of the fluid.

2.4 Attached Mass

Any object moving through a fluid will need to displace the fluid around it to occupy a new position. This means during its motion the object must not only dispense energy to move its own mass, but must also accelerate the fluid around its bulk. This action must increase the kinetic energy of the fluid and the object must do additional work to accomplish this. This way of considering the problem is overly complicated when creating a model, but luckily we can simulate the same effects in an easier way. We can model the extra work done by the object as an extra amount of mass that is "attached" to the object and moves with it through the fluid.

The kinetic energy of this attached mass, traveling at a velocity U, should be given by the following equation:

$$T = \frac{1}{2}\rho I U^2 \tag{11}$$

I is a term that denotes how changes in the velocity of an element of volume of fluid affects the body associated with the attached mass. This is given by the following equation:

$$I = \int_V ((\frac{u_1}{U})^2 + (\frac{u_2}{U})^2 + (\frac{u_3}{U})^2) dV \tag{12}$$

where u_1, u_2, u_3 represent the components of the fluid velocity.

For a sphere, the potential is given by:

$$\psi = -\frac{UR^3}{2r^2} cos(\phi) \tag{13}$$

Converting the velocities to spherical coordinates, we get:

$$u_r = \frac{\partial\psi}{\partial r} \tag{14}$$

$$u_\phi = \frac{1}{r}\frac{\partial\psi}{\partial\phi} \tag{15}$$

Substituting the variables in the previous equation into the equation for I gives the following formula:

$$I_{sphere} = \int_V ((\frac{1}{U}\frac{\partial\psi}{\partial r})^2 + (\frac{1}{Ur}\frac{\partial\psi}{\partial\phi})^2) dV \tag{16}$$

Which then reduces to:

$$I = \frac{2}{3}\pi R^3 \tag{17}$$

We can then find mass, given the density of the fluid, ρ.

$$m_{att} = \frac{2}{3}\pi\rho R^3 \tag{18}$$

This is half of the mass of the liquid displaced by the bubble. Therefore, the attached mass caused by a spherical object moving through a liquid is half the mass of the fluid displaced by the object. This half mass is being pulled or 'dragged along' by the movement of the bubble in the fluid.

2.5 Governing Equation

After combining the various forces associated with this motion and inserting the model parameters into Newton's Second Law, the governing equation of the bubble system is given by:

$$(m + m_{att})\ddot{x} + \dot{m_{att}}\dot{x} = -F(\dot{x}) + (m - \rho V(t))(A\omega^2 \sin(\omega t) + g) \tag{19}$$

m_{att} is the attached mass of the bubble, $\dot{m_{att}}\dot{x}$ is the term associated with the variation of the attached mass, $-F(\dot{x})$ represents the drag force, and the last term is associated with the buoyancy force and the pressure fluctuations in the vibrating liquid.

2.6 Separation of Variables

In a common method used in harmonic analysis, the method of separation of variables is used.

$$x(t) = X(t) + \psi(t) \tag{20}$$

Here, $X(t)$ is the equation of 'slow motion,' or also kwown as the equation of general motion (i.e. whether the bubble moves upward, downward, or is stationary). Ψ is the equation of 'fast motion,' so it describes the fast oscillating terms in the model.

The oscillatory terms are considered the fast terms in the equation of motion. These all have a time average motion of 0, for a uniformly oscillating liquid, and therefore are not accounted for in the general motion of the system.

The system is then separated into the 'fast' and 'slow' equations of motion. The 'fast' solution is shown:

$$(m + m_{att})\ddot{\Psi} = -4\rho R_0^2(Re)(\dot{\Psi}^2 sgn\dot{\Psi} - (\dot{\Psi}^2 sgn\dot{\Psi})) + (m - \rho V(0))A\omega^2 sin(\omega t)) \tag{21}$$

The 'slow' solution is show here as:

$$(m + m_{att})\ddot{X} + \langle F(\dot{X} + \dot{\Psi})\rangle = \gamma\omega^2 \frac{\rho V(0)g}{2}\frac{X}{H_0}(1 - \frac{2}{3}(1 - \frac{m}{\rho V(0)})sin^2(\phi)) - (\rho V(0) - m)g \tag{22}$$

where $\gamma = \frac{\rho H_0 g}{P(0)}$. H_0 is defined as the height of the fluid column.

We then assume that the acceleration term is relatively small and remove the second order terms (both are due to the small size of the bubbles). We consider the mass of the bubble to be relatively small in comparison to the attached mass of the system, i.e. $m << m_{att}$.

In solving for these solutions, it should be noted that this 'slow' solution actually depends on the 'fast' solution. In considering the fast solution, we assume that the amount it oscillates is greater than the amount it moves in a time interval, or that $|\dot{X}| << |\dot{\Psi}|$. We find the fast oscillation is described as follows:

$$(m + m_{01})\ddot{\Psi} = -\rho R_0^2 \Psi_\infty(\dot{\Psi}^2 sgn(\dot{\Psi}) - \langle\dot{\Psi}^2 sgn(\dot{\Psi})\rangle) + (m - \rho V_{b0})A\omega^2 sin(\omega t) \tag{23}$$

Where the above equation dictates how the bubble oscillates rapidly while moving through the fluid. From this, we can find the following equality:

$$sin(\phi) = B^2 \frac{16\Psi_\infty}{2\pi^2 A R_0} \tag{24}$$

Where B, the amplitude, is found to be:

$$B^2 = \frac{2A^2}{x^2 + \sqrt{x^4 + (\frac{16A\Psi_\infty}{\pi^2 R_0})^2}} \tag{25}$$

Now having the fast solution defined, we can then move onwards to the slow solution. In doing so, we define the following:

$$\langle \ddot{\Psi} sin(\omega t) \rangle = -\frac{1}{2}\omega^2 B sin(\Phi) \tag{26}$$

$$\langle \dot{\Psi} cos(\omega t) \rangle = \frac{1}{2}\omega B cos(\Phi) \tag{27}$$

Using these equalities, we can write down the 'slow' solution equation:

$$m_{01}\ddot{x} + \frac{16}{\pi}\rho\Psi_\infty R_0^2 B\omega\dot{x} = \gamma\omega^2 \frac{\rho x g}{2H_0} V_{b0}(1 - \frac{2}{3}\frac{\theta\frac{A^2}{R_0^2}}{2(1+\sqrt{1+\theta\frac{A^2}{R_0^2}}) + \theta\frac{A^2}{R_0^2})} - \rho g V_{b0} \tag{28}$$

where $\theta = \frac{16^2}{\pi^2}\frac{\Psi_\infty^2}{X^4}$

This describes the trajectory of the bubble through fluid, how it generally rises or sinks in the presence of oscillations.

2.7 Averaging Procedure

The separation of variables indicates that the solution is going to be a sum of two time scaled functions represented by the following equation.

$$\langle x(t) \rangle = \langle X(t) \rangle + \langle \Psi(\tau) \rangle \tag{29}$$

Where, $X(t)$ is the slow time scale function (represents the trajectory of the bubble while sinking) and, $\Psi(\tau)$ is the fast time scale function (represents the oscillation of the bubble). Since the fast time scale function $\Psi(\tau)$ is periodic, it has an average value of 0 over one period. On the other hand, the slow time scale function has an average value of itself. Therefore, the whole trajectory of the bubble has an average value that is equal to the average value of the slow time scale function $X(t)$. So, the average equation ends up as the following:

$$\langle x(t) \rangle = \langle X(t) \rangle \tag{30}$$

Thus, we can see that the whole trajectory of the bubble is described by the slow time scale function.

Additionally, we can assume that the acceleration of the bubbles is fairly small, which allows us to find that:

$$\dot{X} = v(\frac{X}{X_0} - 1) \tag{31}$$

Where the v and X_0 are given as follows:

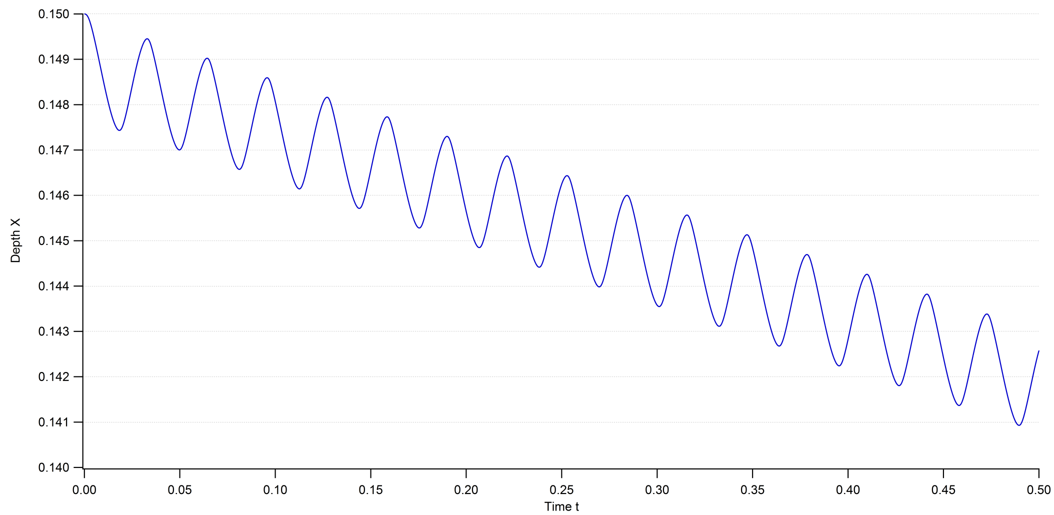
$$v = \frac{\pi^2}{12\Psi_\infty}\frac{gR_0}{B\omega} \tag{32}$$

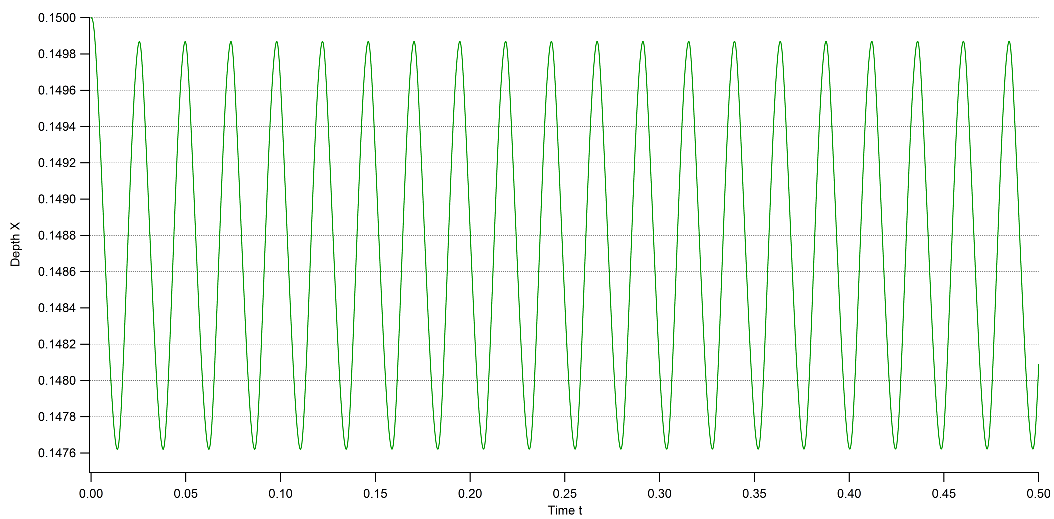
$$X_0 = \frac{2H_0}{\gamma\omega^2}\frac{2(1+\sqrt{1+\theta\frac{A^2}{R_0^2}}) + \theta\frac{A^2}{R_0^2}}{2(1+\sqrt{1+\theta\frac{A^2}{R_0^2}}) + \frac{\theta}{3}\frac{A^2}{R_0^2}} \tag{33}$$

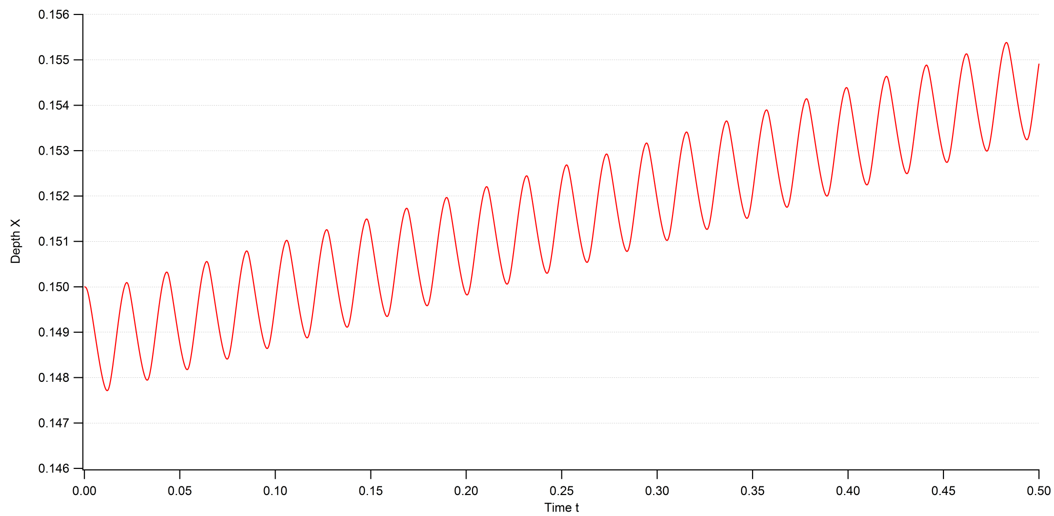
From the velocity equation above, we can find a few regimes of movement. If $X < X_0$, then the bubble will float, if $X = X_0$ then the bubble will remain about motionless, and if $X > X_0$ the bubble will sink. We can see now that the oscillation of fluids will only allow the bubble to sink if specific conditions are met.

3 Stability and Analysis

From the above equation, we have ascertained the various regimes of movement, sinking, stationary, and floating. The question is when each of these situations occurs. The value for X_0 determines where the regimes are located in the fluid volume. This establishes 3 zones: $0 < X < X_0$, $X = X_0$, and $X_0 < X < \infty$. In order to observe these states, we created a computational model that utilizied Verlet integration to produce solutions.

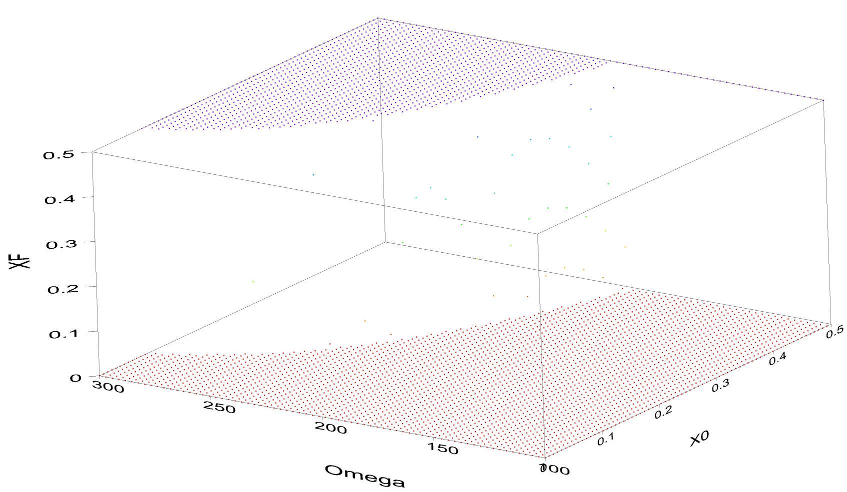






The above graphs show solutions for 3 different frequencies, each beginning at a depth of 0.15 m. The 'equilibrium depth' X_0 is altered by the value of the frequency of oscillation ω.

In order to understand the stability of the regimes, we can plot the values of ω, X_0, and the final depth XF to find the stability of regimes.



We can see the densely populated zones of blue and red give us the regimes of sinking and rising respectively. The space between the red and blue, where $X_0 = XF$ is the sparsely populated zone where bubbles are essentially motionless. The rising and sinking zones are stable regimes, while the motionless area is unstable

There are some conclusions to be drawn about this diagram, the first being that there is no frequency that will cause sinking given a small enough initial depth. Also, the ability to sink only presents itself at sufficiently high frequencies.

4 Plasmonic Nanoparticles

Recent research has shown some interesting applications of plasmonic nanoparticles when in an aqueous solution. When plasmonic nanoparticles are exposed to electromagnetic radiation, an oscillating dipole forms in the material. This causes the nanoparticles to oscillate and heat up rapidly. In water, the intense heating forms bubbles that continue to be heated from the inside. This causes the gas in the bubbles to heat up, and increase the volume of the bubbles. We assume that this process is adiabatic, and that all of the previous assumptions hold.

4.1 Adiabatic Process

Consider steam bubbles generated from incident light on 100 nm gold particles suspended in the fluid. We consider the number of steam particles to be constant over time, with a relatively non-changing pressure inside caused by the balance of the Laplace pressure and the internal pressure. The only effect on the bubble pressure is given by the oscillations. By the ideal gas law, we arrive at the following expression:

$$\frac{P(t)V(t)}{T(t)} = \frac{P(0)V(0)}{T(0)} \tag{34}$$

We let $T(0) = 373.12$ K, as this is the initial temperature of the generated steam. We let the temperature of the bubble increase by the following linear function:

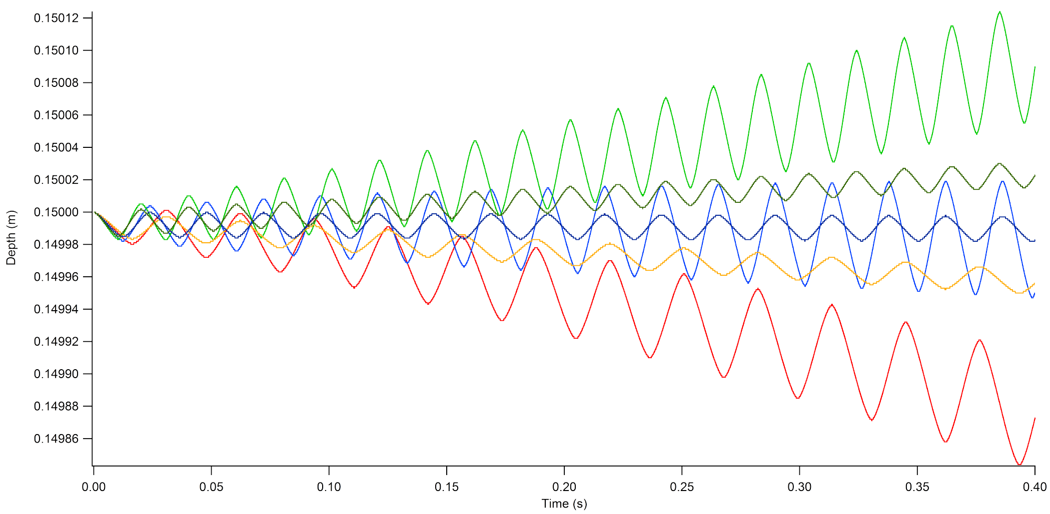
$$T(t) = T(0) + \frac{I_{inc}\sigma t}{4\pi k R_{NP}} \tag{35}$$

Here, I_{inc} is the intensity of the incident light, σ is the cross-section of the nanoparticle, k is the thermal conductivity of gold, and R_{NP} is the radius of the nanoparticle.

We assume the oscillating pressure term as given above, and combine these terms to get the final adiabatic volume term:

$$V(t) = \frac{P(0)V(0)(1 + \frac{I_{inc}\sigma t}{T(0)4\pi k R_{NP}})}{1 + \frac{\rho g x}{P(0)} + \frac{\rho x A\omega^2 \sin(\omega t)}{P(0)}} \tag{36}$$

This produces a changing volume, which we can then input into the prior equations as before. We can then model the solutions using the same method as before.



The above figure shows the comparison of solutions with particle addition and without. When the bubbles contain plasmonic nanoparticles, the amplitude of oscillation increases, and the rate of sinking or rising is increased. This is a valuable result, as it allows us to further control the dynamics of bubbles by introducing gold particles.

5 Conclusion

The dynamics of bubbles in vibrating fluids is an interesting topic, as it defies the common sense observation of bubbles always floating. However, under certain oscillatory solutions we can force bubbles to sink. We find that there exists a specific depth at which bubbles will remain motionless, however, this state is unstable. The stable solutions are when bubbles form below or above this critical depth, and rising and sinking occur.

In the presence of nanoparticles, we can shine light and cause the bubbles' volume to increase with time. This causes the amplitude of oscillation of the bubble to increase, and the rate of sinking/rising to increase. This was verified via computational modeling, which demonstrated these effects.

6 Sources

- Brennen, C. E., comp. "A Review of Added Mass and Fluid Inertial Forces." (1982): Naval Civil Engineering Laboratory. Web. 8 Mar. 2014. authors.library.caltech.edu/233/1/BRE052.pdf
- Falkovich, G. Fluid Mechanics: A Short Course for Physicists. Cambridge: Cambridge UP, 2011. Print.
- V.S. Sorokin, "Motion of a Gas Bubble in Fluid Under Vibration," (201): Springer. Web.
- Techet, A. "Object Impact on the Free Surface and Added Mass Effect." (2005): Massachusetts Institute of Technology. Web. 8 Mar. 2014. 'web.mit.edu/2.016/www/labs/$L01_{A}dded_{M}ass_{0}50915.pdf$'