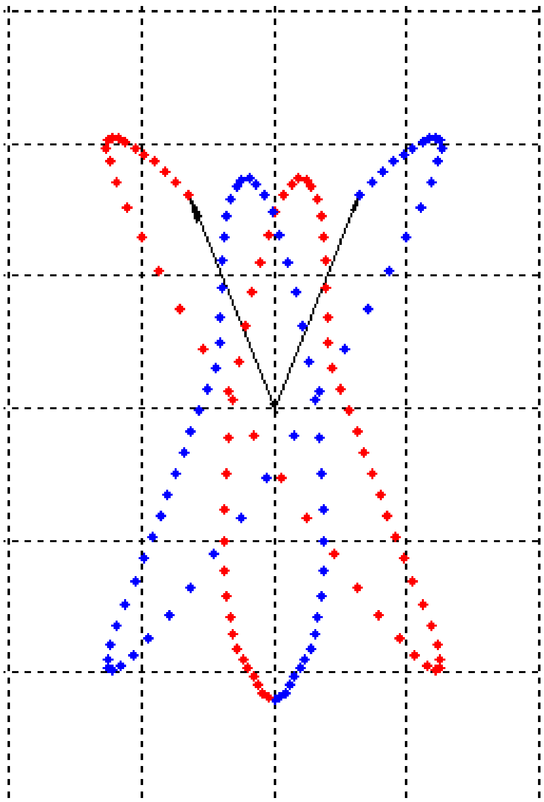


LARGE-AMPLITUDE DYNAMICS OF THE ELASTIC PENDULUM

A group project by Corey Zammit, Nirantha Balagopal, Qisong Xiao, Zijun Li and Shenghao Xia

ABSTRACT

A numerical computer model that shows the motion of a mass freely swinging from an elastic band as a representation of a bungee jumper was improved by comparison with data from a physical model and demonstrated to accurately account for the nonlinearity of the elastic response of a resistance band for approximately 3 seconds of motion.

MATH MODELING FINAL REPORT

MATH 485; Dr. Ildar Gabitov; Mentor: Joe Gibney

1 An Introduction to the System Considered

The classic “swinging spring” is the problem that the team set out to investigate, and the topic is addressed in detail in the midterm report. The system is simply a mass at the end of a spring swinging freely from a pivot; the question is what does its motion behave like in different regimes? Among the details addressed are the vast limitations that the assumptions that must be made present on the process of creating a physical model, especially for regimes of large amplitudes. The primary concern with considering the “swinging spring” system is the entirely necessary assumption that the spring does not bend, which is acceptable when small amplitudes are considered because the spring never enters compression. However when large amplitudes are considered this assumption is impossible to accept; because all springs bend when they are put into compression without constraint. Furthermore any constraints that could be imposed on a physical system would inherently create more friction. This cannot be accounted for mathematically in large amplitudes and would have some effect on the assumption that the spring is massless and does not interact with the bob for all purposes other than to create a reaction force from extension and compression.

The adapted system that the team set out to model is one with fewer assumptions. The system considered is a bungee jumper. The bungee jumper is small in length compared to the length of a bungee cable. Since the cable is attached near the bungee jumper’s center of mass, the bungee jumper is effectively a point mass. Also, the bungee jumper is greater in mass than the bungee cable, and in the case of the physical model that was constructed this is certainly true (this will be discussed in **section 5**). For this reason we test the validity of assuming that the bungee cord is massless. Also with this same reasoning we will assume that the bungee cord does not interact with the bob in any manner other than to create a reaction force under extension; of course a bungee cord has no reaction when it is not extended from its relaxed length. One more assumption that was made for the numerical model is that there is no loss of energy in the system, and the reason for assuming this becomes obvious when we derive the equations of motion in **section 2**. Assuming that there is no friction is of course the largest downfall of the numerical model yet there hope is that the behaviors are similar for an observable period of time.

2 Derivation of the Equations of Motion

Creating a numerical model requires equations of motion that are customized to our experiment. From research made available to us we have seen the derivation of a system of three ordinary differential equations from the Lagrangian equation. This derivation assumes a linear relationship between the extension of the spring and the reaction force. This derivation was summarized and justified in the midterm report and has the form $L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{k}{2}(r - l_0)^2 - mgz$ where $r = \sqrt{x^2 + y^2 + z^2}$, m is the mass of the bob, g is the gravitational acceleration constant, and l_0 is the characteristic relaxed length of the spring. The first term in the equation is the kinetic energy of the mass, and the last term in the gravitational potential energy of the mass. These terms are consistent with the system that we are considering; the term that must be adapted is the second term: the elastic potential energy.

Since large amplitudes are considered, the possibility that the bungee cord extends beyond the range that it behaves linearly must be considered. From preliminary tests of some household resistance bands it was determined that a fit of the form $F_{elastic} = a(r - l_0)^3 + b(r - l_0)$ would serve well. The procedure by which the tests were performed are discussed in **section 4**. The fit of that form was contrived based on the fact that the behavior of the spring was nonlinear and an odd fitting function was required for the end behavior and type of symmetry.

Entering this fit in the Lagrangian starts with integration. The potential energy from extending the resistance band, here referred to as $V_{elastic}$; is integrated over an indefinite displacement to get:

$$v_{elastic} = \frac{a}{4}(r - l_0)^4 + \frac{b}{2}(r - l_0)^2$$

Now the Lagrangian is:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{a}{4}(r - l_0)^4 - \frac{b}{2}(r - l_0)^2 - mgz$$

And differentiating with respect to time assuming L is constant yields:

$$0 = m(\dot{x} \cdot \ddot{x} + \dot{y} \cdot \ddot{y} + \dot{z} \cdot \ddot{z}) - a(r - l_0)^3 \cdot \frac{x \cdot \dot{x} + y \cdot \dot{y} + z \cdot \dot{z}}{\sqrt{x^2 + y^2 + z^2}} + b(r - l_0) \cdot \frac{x \cdot \dot{x} + y \cdot \dot{y} + z \cdot \dot{z}}{\sqrt{x^2 + y^2 + z^2}} - mg \cdot \dot{z}$$

Substituting $r = \sqrt{x^2 + y^2 + z^2}$, dividing through by m, and factoring terms of $\dot{x}$, $\dot{y}$, and $\dot{z}$ yields:

$$\begin{aligned}
&0 = \cdots \\
&+\dot{x}\left(\ddot{x} - \frac{a}{m} \cdot \frac{(r - l_0)^3}{r} x - \frac{b}{m} \cdot \frac{r - l_0}{r} x\right) \\
&+\dot{y}\left(\ddot{y} - \frac{a}{m} \cdot \frac{(r - l_0)^3}{r} y - \frac{b}{m} \cdot \frac{r - l_0}{r} y\right) \\
&+\dot{z}\left(\ddot{z} - \frac{a}{m} \cdot \frac{(r - l_0)^3}{r} z - \frac{b}{m} \cdot \frac{r - l_0}{r} z - g\right)
\end{aligned}$$

So now we have a system of ordinary differential equations that describe Cartesian acceleration as a function of Cartesian coordinates with 5 parameters. The equations of motion are:

$$\begin{cases}
\ddot{x} = -\left(a \cdot (r - l_0)^3 + b \cdot (r - l_0)\right) \dfrac{x}{m \cdot r} \\
\ddot{y} = -\left(a \cdot (r - l_0)^3 + b \cdot (r - l_0)\right) \dfrac{y}{m \cdot r} \\
\ddot{z} = -\left(a \cdot (r - l_0)^3 + b \cdot (r - l_0)\right) \dfrac{z}{m \cdot r} - g
\end{cases}$$

Though the system is not integrable and no analytic solutions exist, the dynamics can be studied numerically using powerful numerical integration tools available with MATLAB. The numerical methods applied to the system are discussed in **section 3**.

3 Numeric Simulations Using MATLAB

MATLAB has a variety of tools that can be used for numerical integration of different types of ODE's. This system is also a very permitting case, so the creation of a numerical model was simple. Before discussing the function and results of the numerical model, we will define how we view the system in the numerical model:

Parameter:	*Meaning:*
$\boldsymbol{m}$	Mass of the bob, measured in pound-mass
$\boldsymbol{l_0}$	The maximum distance that the center of mass of the bob can be from the pivot before the reaction of the elastic starts acting on the bob. The units used are feet.
a, b	Coefficients of the elastic response curve of the form $F = a(r - l_0)^3 + b(r - l_0)$
g	The gravitational acceleration constant taken here to be 32.2 ft/s^2

For numerical integration, initial conditions must be selected. Since velocity is not present in the equations of motion, though the position is dependent on the velocity, one must specify initial velocities to analyze the motion of the system; and since the acceleration is dependent on the position per the equations of motion, we need only specify six numbers to begin the process of numerical integration:

Datum:	*Meaning:*
x_0	The x −projection of the initial position
$\dot{x}_0$	The x −projection of the initial velocity
y_0	The y −projection of the initial position
$\dot{y}_0$	The y −projection of the initial velocity
z_0	The z −projection of the initial position
$\dot{z}_0$	The z −projection of the initial velocity

For convenience, and since only motion in the x-z plane is discussed in this report: figures will refer to these initial conditions in a sort of vector notation of the form:

$$\boldsymbol{X_0} = (x_0, \dot{x_0}, z_0, \dot{z_0})$$

The coordinate system is defined such that the origin is coincident with the pivot and with the positive z- axis pointing upward. **Figure 1** shows an arbitrary set of initial conditions to point out a few features of the simulation. In the figure the vertical arrow shows the relaxed, characteristic length "l_0" which is not to be mistaken with the equilibrium length of the system with a mass attached even though the vertical orientation may be misleading in that sense. The other arrow represents the elastic band. It roughly points to the end of the trajectory that represents the position at the time the simulation stops, and while the program is running the vector shows the length of the elastic band at the time compared to its characteristic length. Starting from $(-1,0,-1,0)$ notice that the motion is entirely vertical until the trajectory turns blue when the path is influenced by the elastic band. As you can see this is also when the distance from the pivot becomes greater than the vector representation of l_0. In **section 4** we will discuss the experimental design to test the features of this program.

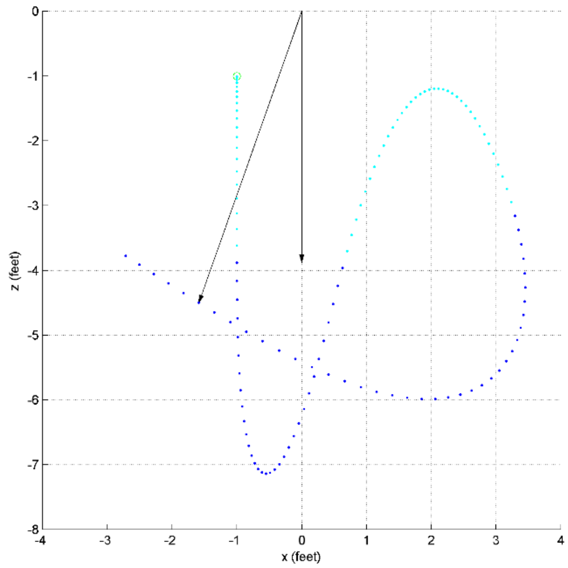


Figure 1; an arbitrary simulation starting from (-1,0,-1,0); cyan represents free fall and blue represents trajectories with an elastic response. 50 dots per second.

4 Experimental Design

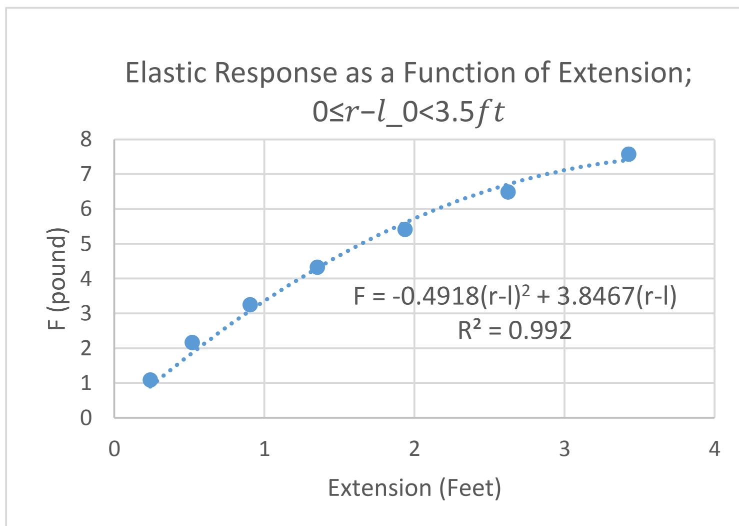
The first phase in the experiment design was deciding what type of system could be employed. Obviously dealing with the magnitudes of an actual bungee jump would be difficult; also the typical, vertical dynamics of a bungee jumper are trivial for this model. Our first idea (discussed in the midterm report) was to connect springs in series, particularly for the reason that we believed springs would behave more linearly than bungee cables. However in systems that the team tested initially –particularly those with a larger ratio of the approximated spring constant to the bob mass: $\frac{k}{m}$, the jostling of the individual springs caused too much divergence from the predicted path and the motion was too "busy" to compare to the model. Moreover in smaller ratios it was obvious that the fit for reaction force related to extension was not exactly linear as anticipated. For these reasons the next (and also more obvious) experimental design was conceived.

Figure 2; a generic photograph of a similar resistance band to those used in our experiment.

Household resistance bands (**figure 2**) were tested for their elastic response function. This relationship was decidedly nonlinear, though this is the cost of taking on the consideration of large amplitudes. Then the type of mechanisms this might be uses for connecting bobs were considered. An effective solution to this was tightly wound string and hand wraps. Also it was noted that the spring function should be tested within the range of the possible extension of these prospective bobs, starting from a maximum initial z-coordinate of 0, and with $r - l_0$ not exceeding 1 foot.

The outlined procedure for collecting the data was simple but done carefully:

1. The characteristic length of the resistance band was measured. This is l_0.
2. The spring is hung vertically and allowed to come to equilibrium with a small mass attached.
3. The displacement from the characteristic length is measured and recorded.
4. This is repeated with increasing mass six times within the aforementioned range.



Once the data is collected; the experimental dependent variable is switched to be considered the independent variable so force (the weight of the mass attached) as a function of extension (the equilibrium position less l_0) is graphed. The fitting process was nearly ideal because it fit perfectly to a quadratic function in every case. We say this is nearly ideal because the function was limited by the need to avoid over fitting the data. With an odd polynomial of the form mentioned in **section 2** an R^2 value of 0.999 in some cases was seen, which is a sign that a cubic fit may have been over-fitting the data. On the other hand a quadratic polynomial fit the data nearly perfectly as well in every case, so it was decided to use the lower degree polynomial to run our simulations for three reasons:

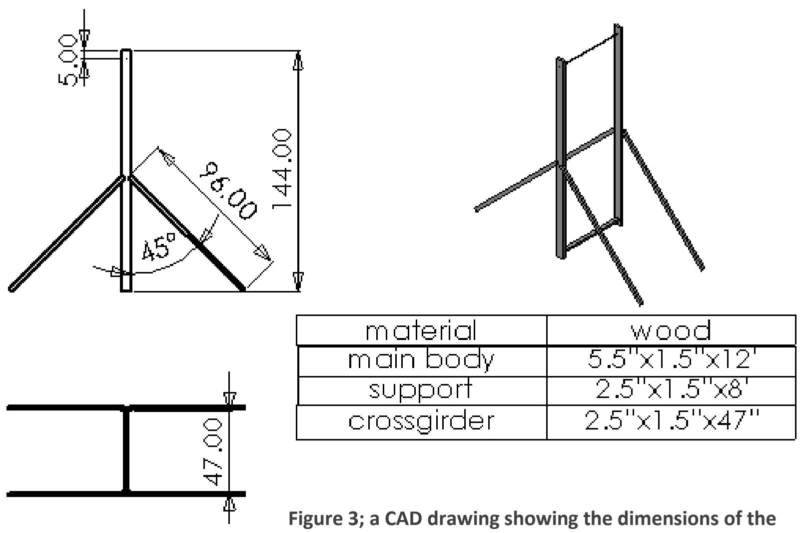
1. There is no need for any type of symmetry about the point of zero extension since the model of an elastic band excludes a compression force.
2. The data that we fit more than spanned the range for extension considered in our experiment, so no concerns involving the concavity or end behavior of our polynomial fitting function must be considered.
3. In some cases, the numerical simulations involving cubic terms caused errors.

Therefore when experiments were recorded, MATLAB figures were created using the best polynomial fit of the lowest degree, which was in every case a quadratic fit. The system for a spring function of quadratic form is:

$$
\begin{cases}
\ddot{x} = -\left(a \cdot (r - l_0)^2 + b \cdot (r - l_0)\right) \dfrac{x}{m \cdot r} \\
\ddot{y} = -\left(a \cdot (r - l_0)^2 + b \cdot (r - l_0)\right) \dfrac{y}{m \cdot r} \\
\ddot{z} = -\left(a \cdot (r - l_0)^2 + b \cdot (r - l_0)\right) \dfrac{z}{m \cdot r} - g
\end{cases}
$$

5 The Physical Model

A physical model was realized first by developing a support for the pivot of the pendulum considered in 2-dimensions. The support was created with enough clearance to allow the bob to swing from the resistance bands in the amplitudes that were to be considered without hitting the ground. Supplies were collected after a CAD model of the pivot support system was created. The support was created to restrict rocking of the pivot in the x-direction and such that the pivot was not able to bounce along the z-axis. The CAD drawing of the pendulum pivot support is shown in **figure 3**. The cross girder and cross-brace supports were fixed to the vertical support at a 45° angle and forced into the ground with boulders to achieve maximum frictional stabilization. The rod that used as pivot was driven through the main body tightly into ¼ inch pilot holes.



material	wood
main body	5.5"x1.5"x12'
support	2.5"x1.5"x8'
crossgirder	2.5"x1.5"x47"

Figure 3; a CAD drawing showing the dimensions of the support system for the pendulum pivot.

Figure 4; the resistance band set that was used for the physical model.

The articulation of the resistance band with the pivot was minimal as the rod we used as a pivot was grease-coated steel and the end of the resistance band was a metal hook that fit snugly, but not tightly, around the rod. A stock image of the exact resistance band set that was used for this experiment is shown in **figure 4**.

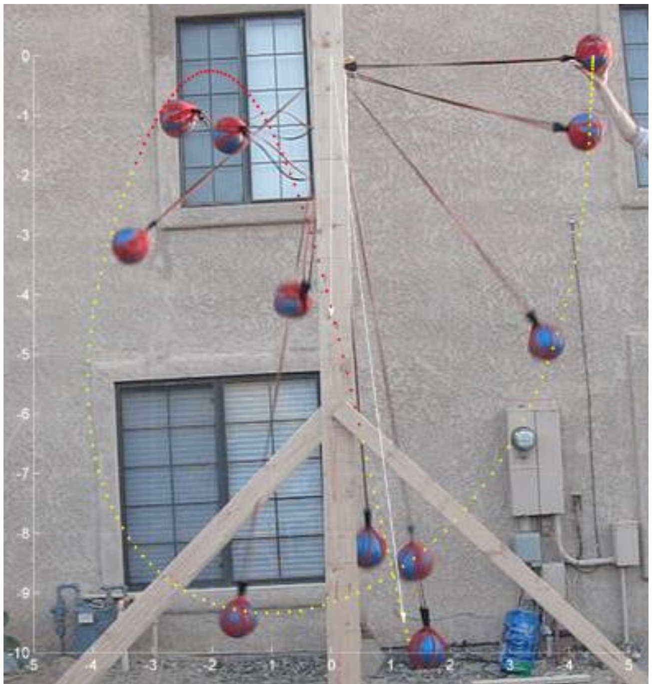


Figure 5; frames separated by 0.25 second intervals of a 10 pound ball swinging with corresponding numerical simulation superimposed. $X_0 = (l_0, 0, 0, 0)$, where $l_0 = 4.354$ feet, $a = -0.8028$ and $b = 8.5234$

A high speed camera was provided by the course instructor, Dr. Ildar Gabitov and the University of Arizona Department of Mathematics. Using video recorded at 120 frames per second, data was collected from an angle set normal to the x-z plane and centered approximately at the middle of the trajectories that were considered. Figures of the numerical simulations for the corresponding parameters and initial conditions were superimposed with square axes on top of overlaid frames of the video at ¼ second intervals. **Figure 5** shows the one such scenario with a 10-pound medicine ball. The simulation was not specially selected in any way; the initial conditions were chosen based on the simple means of consistency and repeatability –i.e. it is easy to hold a ball horizontally with no extension and drop it with no initial velocity. The parameters involving the length of the resistance band and the mass of the medicine ball were simply recorded as accurately as possible and the fitting of the spring function was done with all precision available; and the numerical simulation exceeded expectations on the first try.

Figure 6 shows the same respective initial conditions with a different resistance band and a three-pound dumbbell instead of the medicine ball. It was possible to attach the resistance band to the dumbbell at its center of mass unlike the case with the medicine ball, and so the characteristic length of the resistance band is slightly shorter.

An additional technique used to collect data on the experimental trajectories of the system was long exposure photography. A long exposure photograph is a still frame photograph where the shutter stays open long enough to collect an exposure in low-light situations. When intense light is thrown across the field of view during an exposure it leaves a streak of light showing the path of the light source. We employed this technique by attaching a bright LED head lamp to the bob at night time and repeated a couple of the experiments that we performed during the day time. The results were very illuminating.

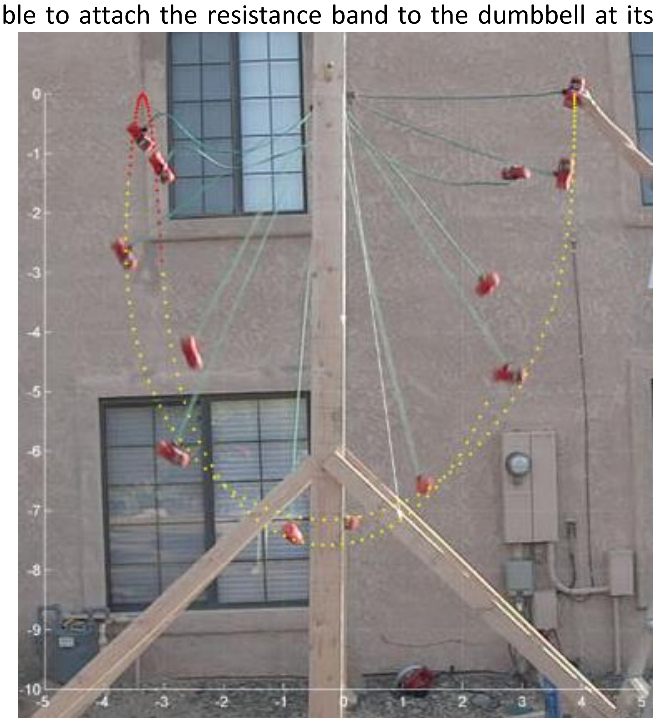


Figure 6; frames separated by 0.25 second intervals of a 3 pound dumbbell swinging with corresponding numerical simulation superimposed. $X_0 = (l_0, 0, 0, 0)$, where $l_0 = 3.875$ feet, $a = -0.3807$ and $b = 3.589$

Figure 7 shows the same experiment demonstrated in **figure 5** in a different light. The exposure on the photograph is about 5 seconds. Unfortunately the ball is rotating during the swinging motion so the head lamp becomes eclipsed sooner than we would have liked, but nevertheless it is pleasing to see that it shows the motion in agreement with the figure overlays and again with the numerical simulation. Since the ball is continuously rotating one might allow a margin of error in this overlay equal to the radius from the center of mass of the ball to the end of the head lamp. That radius is approximately 4 inches.

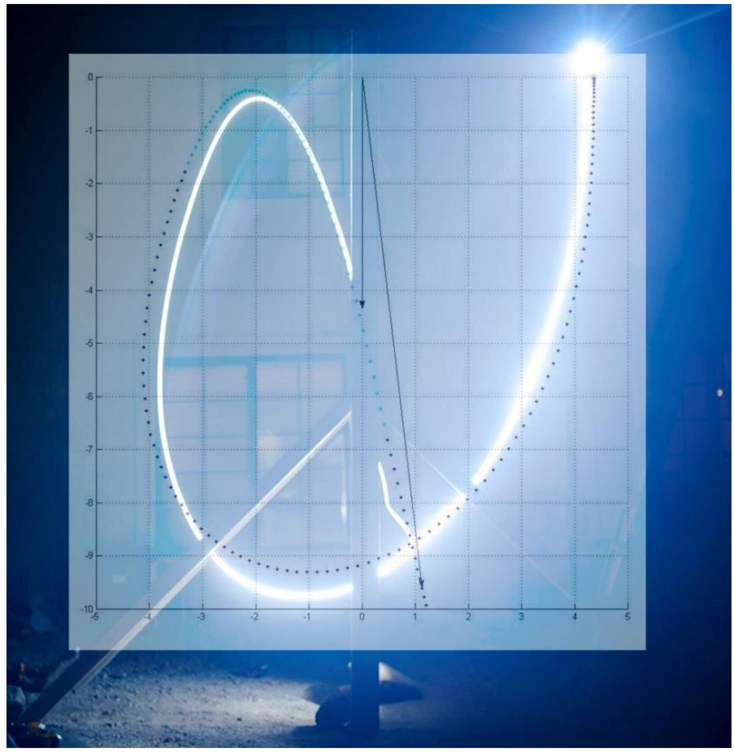


Figure 7; using a long exposure photograph the experiment outlined in figure 5 is shown with a continuous trajectory.

6 Experimental Data Analysis

One out of every thirty frames was taken from the high speed video and the position of the bob was recorded. This gave a course data representation (an approximate position every ¼ second) of the bob's position versus time. Graphs of data collected from the experiment shown in the **data appendix**. Estimating the positions was an arduous process and the velocities were even tougher. For this reason this data was not used to calculate total energy of the system versus time. As a word of caution when interpreting the graphs: The decay of the amplitude in the x-t and z-t projections does not necessarily represent the loss of energy in the system because energy can translate to different axes. The illusion in the z-t graph that the amplitude is decreasing monotonically should be interpreted through this lens.

The v_x and v_z tell a similar story, if carefully inspected together it might show that the amplitudes of velocity, $\sqrt{v_x^2 + v_z^2}$, is decreasing though time though the analysis proves to be more complicated than this calculation as energy can be hidden in the extension of the resistance band as well as in the z-displacement.

v_z-t is a valuable representation of how the dynamics of an elastic band differs from the classic swinging spring problem. The elliptical shape shows the asymmetry of the reaction force from the elastic band when it is working with gravity versus against gravity. You can see that the curve is looser when the resistance band is not activated (on the right) versus when the resistance band is activated (on the left). One would expect this curve to have a taste of symmetry when the elastic band forcing is symmetrical about its characteristic length.

7 Conclusion & Discussion

The experiments reported were not the only experiments performed. Many of the experiments performed were consistent with the numerical model and many served to improve the numerical model. With the large success of this simplified program that was created one question is raised: What is the model useful for? The most apparent use of the model is to demonstrate the trajectories of a bungee jumper or perhaps any other mass swinging from an elastic band. This is an invaluable preliminary step in ensuring the safety of experimentation. For many reasons we call this only a preliminary step –that is it should be taken with one's better judgment and research. One of these reasons in that the assumptions made in this model limit its accuracy. Since the energy is held constant in this application someone might facilely assume that the program shows the maximum magnitude possible for a certain direction of extension, however due to the potentially chaotic behavior of the system the change in the trajectory could lead to extension in an unexpected direction. Another scenario where the model could differ in practice is when bungee jumpers are in free fall. The weight of the chord (around 0.4 pounds/foot) folded over itself can cause gravitational acceleration greater than g, or in almost every case the gravitational acceleration during free fall will be *different* from g when the chord is not in close enough accordance with the assumptions in the model. Therefore an abundance of caution must be made when using a simplified model such as this one. For these reasons the use of the program might just be to get an idea of the qualitative trajectories that can be realized when trying to design a thrilling regime of a bungee jumping experience. This type of simulation could decrease the amount of money spent on prototyping and therefore improving the model for a more specific application could be a useful investment.

8 References & Acknowledgements

Thanks to our mentor Joseph Gibney for getting us started on the MATLAB program and the derivations of equations of motion.

Thanks to our instructor Dr. Ildar Gabitov for lending us a high speed video camera to take important footage of our physical model.

Thanks to Dr. Peter Lynch at the University College Dublin, Director of the UCD Meteorology & Climate Centre, for his research on the dynamics of the elastic pendulum, and for emailing his M-file and allowing us to include video of its display of the fast oscillations of the dynamic pendulum in our midterm presentation.

• 1: Lynch, Peter, 2002: The Swinging Spring: a Simple Model for Atmospheric Balance, Proceedings of the Symposium on the Mathematics of Atmosphere-Ocean Dynamics, Isaac Newton Institute, June-December, 1996. Cambridge University Press

• Craig, Kevin: Spring Pendulum Dynamic System Investigation. Rensselaer Polytechnic Instititute.

• Fowles, Grant and George L. Cassiday (2005). Analytical Mechanics (7th ed.). Thomson Brooks/Cole.

• Holm, Darryl D. and Peter Lynch, 2002: Stepwise Precession of the Resonant Swinging Spring, SIAM Journal on Applied Dynamical Systems, 1, 44-64

• Lega, Joceline: Mathematical Modeling, Class Notes, MATH 485/585, (University of Arizona, 2013).

• Lynch, Peter, and Conor Houghton, 2003: Pulsation and Precession of the Resonant Swinging Spring, Physica D Nonlinear Phenomena

• Taylor, John R. (2005). Classical Mechanics. University Science Books

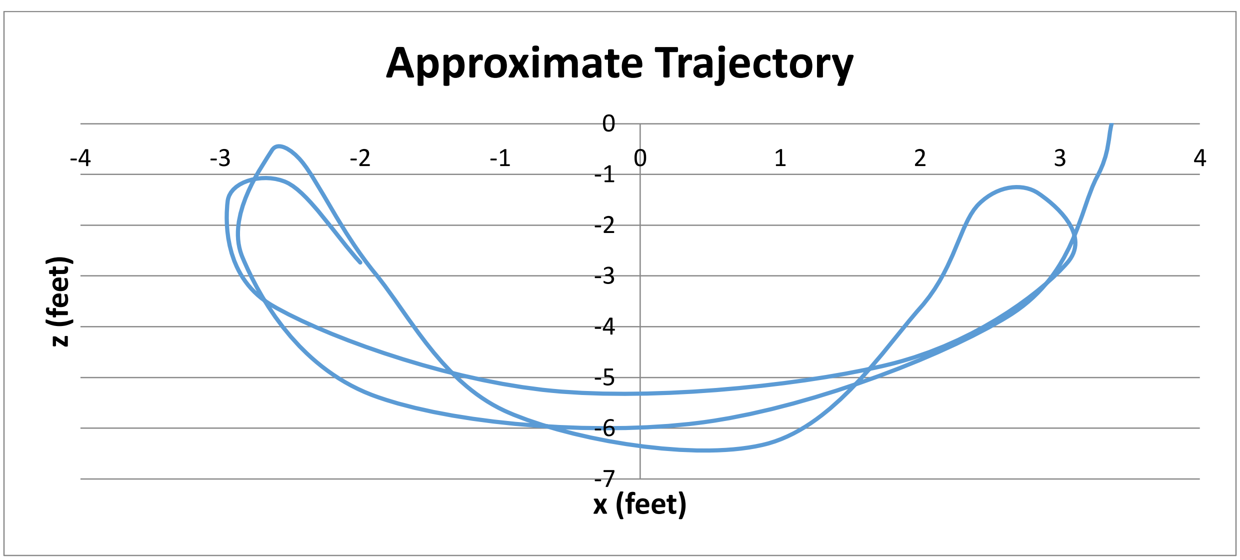
• Thornton, Stephen T.; Marion, Jerry B. (2003). Classical Dynamics of Particles and Systems (5th ed.). Brooks Cole.

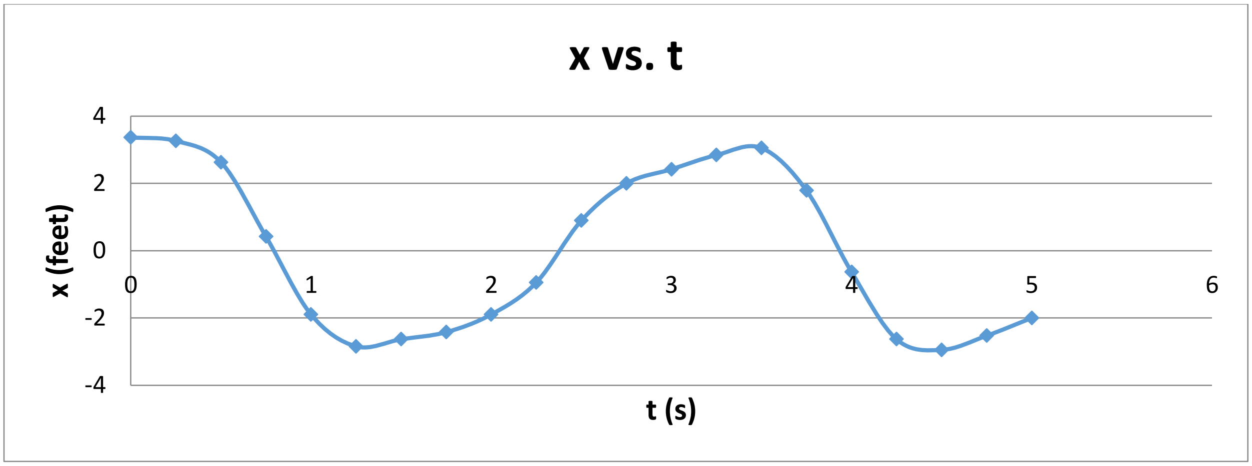
• Vitt, A and G Gorelik, 1933: Oscillations of an Elastic Pendulum as an Example of the Oscillations of Two Parametrically Coupled Linear Systems. Translated by Lisa Shields, with an Introduction by Peter Lynch. Historical Note No. 3, Met Éireann, Dublin (1999)

• Walker, Jearl (2011). Principles of Physics (9th ed.). Hoboken, N.J. : Wiley.

• Lynch, Peter, 2002:. Intl. J. Resonant Motions of the Three-dimensional Elastic Pendulum Nonlin. Mech., 37, 345-367.

Data Appendix

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t (s)
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-4
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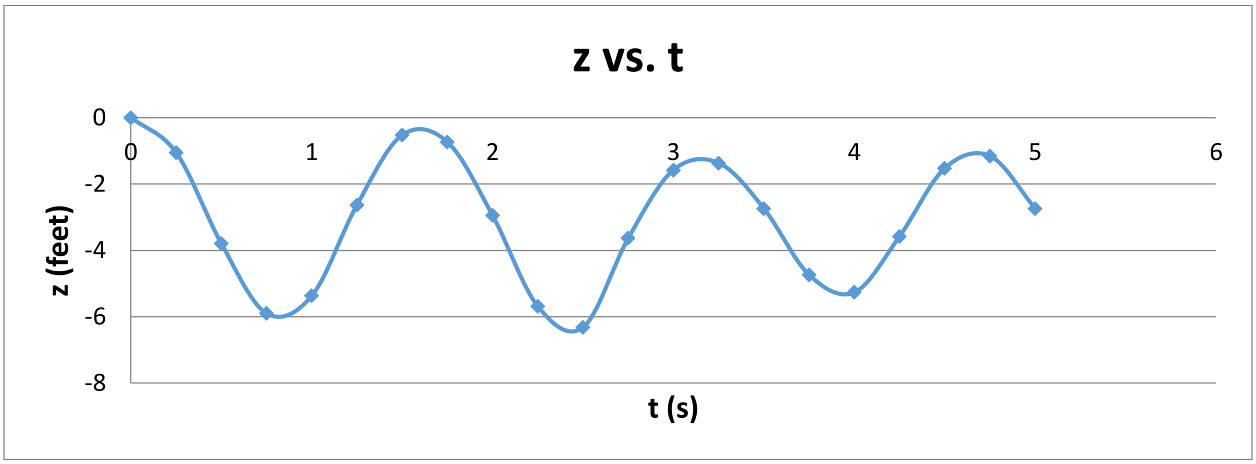
4
5
6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z (feet)
t (s)
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-4
-6
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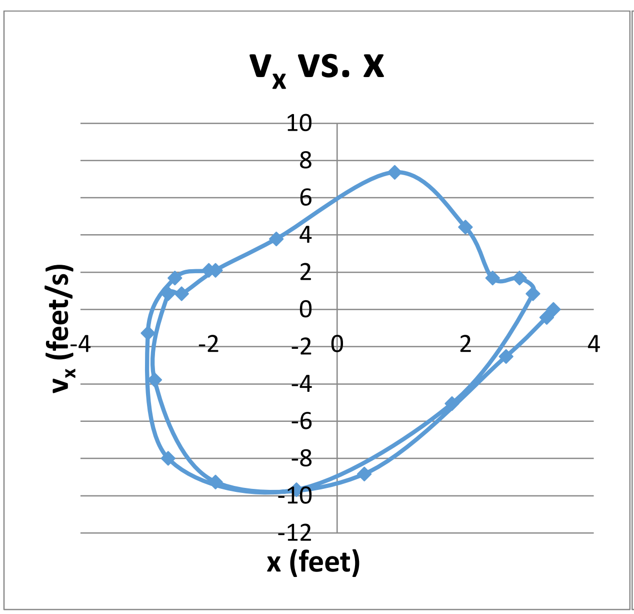
v$_x$ vs. x
v$_x$ (feet/s)
x (feet)
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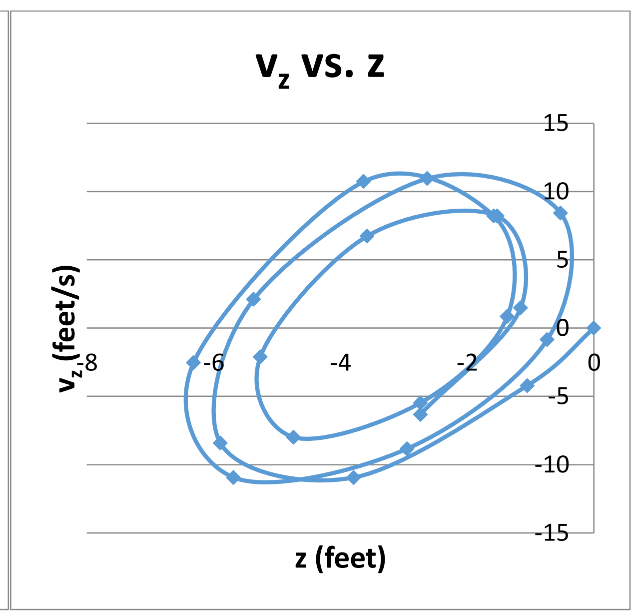
-2
0
2
4


v$_z$ vs. z
v$_z$ (feet/s)
z (feet)
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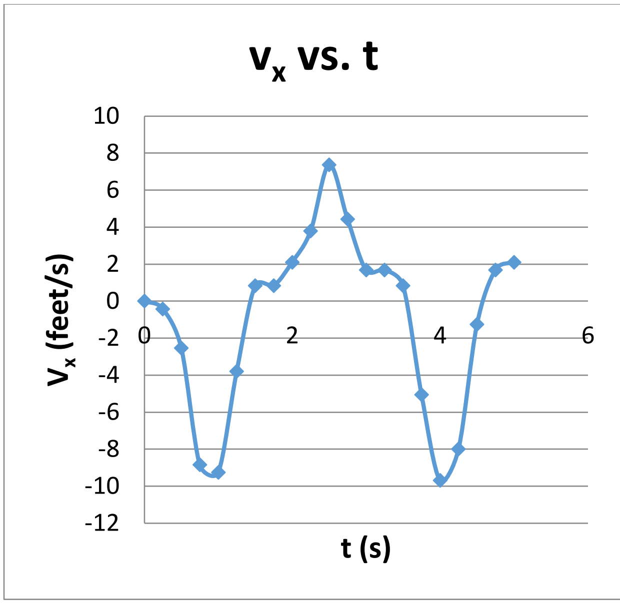
v_x vs. t
V_x (feet/s)
t (s)
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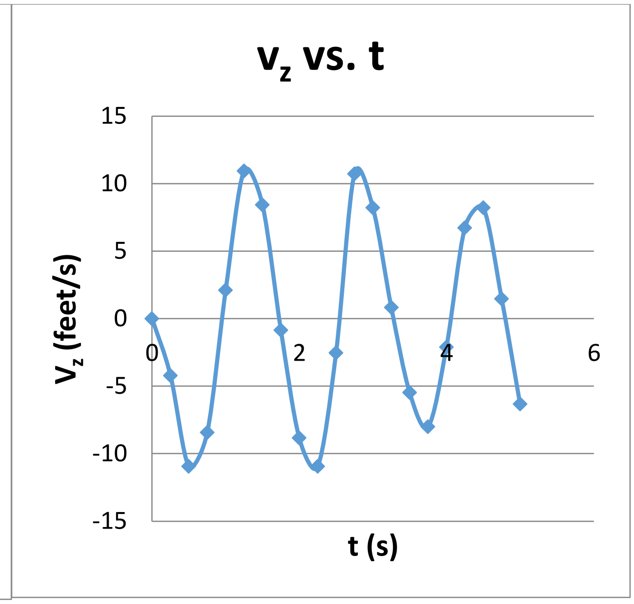
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