Stability Analysis of Pendulum with Vibrating Base

Mentor: Dr. Ildar Gabitov

Thomas Bello

Emily Huang

Fabian Lopez

Kellin Rumsey

Tao Tao

May 6th, 2014

Abstract: A simple pendulum can be unstable at the inverted position, however, it has long been known that adding a vibrating base can change the stability—making it stable at that particular position. Our analysis explores this unusual phenomenon by separating the "fast" and "slow" motion, introducing effective potential, and using the averaging technique.

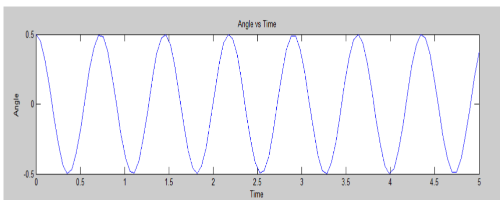
University of Arizona

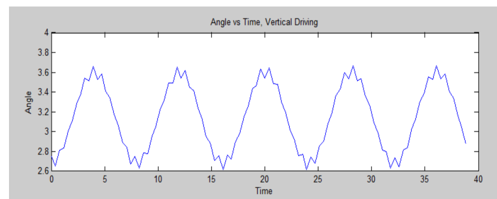
Table of Contents

1. INTRODUCTION

1.1 Simple vs. Vibrating Pendulum

People maybe well acquainted with simple pendulum problems. It is stable downward vertically, and unstable at inverted position. However, when adding a vibrating base on the pivot of the simple pendulum, the system seems to be stable at the inverted position. Simple pendulum swings in a smooth motion (see figure 1a) and is often modeled as $\frac{d^2\theta}{dt^2} + glsin\theta = 0$, where g is the gravitational acceleration, l is the length of the pendulum, and θ is the angular displacement about downward vertical. However, adding a vibrating base would make the motion no longer smooth. It creates small but rapid oscillations on top of the swing, which adds difficulties in the modeling (See figure 1b).





1.2 Background

This subject, pendulum with vibrating base, is in fact well explored by scholars and scientists in the history. In 1908, a scientists call A. Stephenson first questioned that upper vertical position of the pendulum might be stable when the driving frequency is fast. No one could scientifically answer this unusual and counterintuitive phenomenon until almost fifty years later, a Russian physicist Pyotr Kapitza successfully analyzed the stability of this system by separating the motions into fast and slow, and introduced a new concept called Effective Potential.

2. OBJECTIVES

During the midterm, we have derived the Equation of Motion, Lagrangian, and the effective potential of the vertical angle position. Now, we are interested in deriving the equations and analysis stationary solutions of any arbitrary angle position of the system. Our goals are:

1) Derive the Lagrangian for the arbitrary angle
2) Find the Effective Potential using the Averaging technique
3) Analyze the stability at each stationary position
4) Perform an experiment to compare actual results with theoretical findings
5) Error analysis to explain the differences

3. VARIABLES

d_0: Amplitude of base oscillations

ω: Frequency if base oscillations

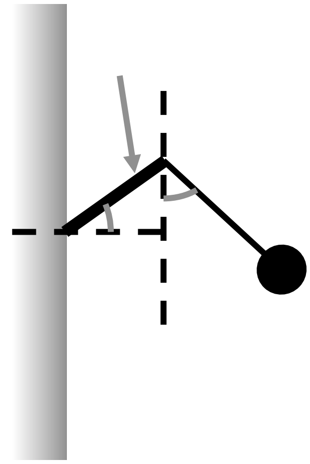
l: Length of the pendulum

θ: Counter clockwise angular displacement of pendulum

g: Gravitational constant

K: Kinetic energy

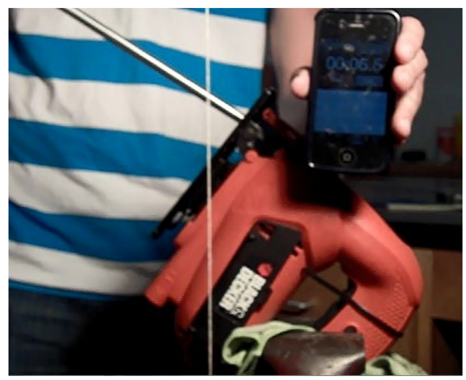
U: Potential energy



4. ARBITRARY ANGLE

4.1. Equation of Motion

Referring to the figure below, we separate the pendulum to x and y direction. The horizontal x-axis is defined as the positive to the right and the y-axis is defined as the positive to the downward. We also assume the gravitation is pointing downward and angles are zero in the y direction with positive in the counterclockwise direction.



The following equations are derived according to the x and y directions in an arbitrary angle:

- $x = lsin(\theta) + d_0 \cos(\omega t) \cos\, \phi$
- $y = l - lcos(\theta) + d_0 \cos(\omega t)\, sin\phi$

where $\sin(\omega t)$ is the motion of the vibrating base and ϕ is the counterclockwise angle of base.

Then, the velocity in each direction will be:

- $v_x = l\dot{\theta} \cos(\theta) - d_0\omega \sin(\omega t) \cos\, \phi$ (1)
- $v_y = l\dot{\theta}\sin(\theta) - d_0\omega \sin(\omega t)\, sin\phi$ (2)

The velocity in each direction is the fundamental input for the later analysis in this project.

4.2 Lagrangian and the Euler-Larange Equation

Now that we have derived Kinetic and Potential energy in terms of the variables for our system, we will use the Lagrangian to further analyze the motion of the Pendulum with a Vibrating Base. Where Kinetic Energy is K, and Potential Energy is U, the Lagrangian (L) is defined as follows:

$$L = K - U \tag{3}$$

4.2.1 Why Lagrangian

The Lagrangian is very useful for us because, in a sense, it contains everything that we need to know about the motion of our system. The Lagrangian also demonstrates a very interesting principle known in Physics as the Principle of Least Action. This principle basically states that in nature, a system will always act in a way such that the 'Action' of the system is minimized. It happens that the Action of the system is also defined as the area under the Lagrangian vs Time curve, and the Principle of Least Action states that the mechanics and motion of the system will be such that this area is at a minimum. The importance of this will become clearer when we introduce the Euler-Lagrange Equation.

4.2.2 Derivation

For now, we will derive the Lagrangian for the Pendulum with a Vibrating Base. Recall the expressions we derived for Kinetic and Potential Energy. By inserting these into (3), we get the following expression for the Lagrangian.

$$L = \frac{1}{2}m(l^2\dot{\theta}^2 + ld_0\omega\dot{\theta}\, sin(\omega t)\cos(\theta - \phi) + mglcos\theta + gd_0cos(\omega t)sin\phi \tag{4}$$

This certainly is the Lagrangian, but we realize that it would be convenient if we had something a little bit simpler to work with. Fortunately, we can take advantage of two properties of the Lagrangian to greatly simplify our result. We can realize that for our purposes:

i) The Lagrangian does not depend on constants.

$$L \equiv aL$$

ii) The Lagrangian does not depend on functions of only time.

$$L \equiv L + f(t)$$

By pulling a length term and a mass term out of our Lagrangian (4), and by eliminating everything that is purely a function of time, we are left with the following expression for the Lagrangian.

$$L = \frac{1}{2}l\dot{\theta}^2 + d_0\omega\dot{\theta}\, cos(\theta - \phi)sin\omega t + gcos\theta \tag{5}$$

With a little bit of further manipulation, keeping in mind that φ is a contant, dependence on $\frac{d\theta}{dt}$ can be eliminated. Thus (5) becomes:

$$L = \frac{1}{2}l\dot{\theta}^2 + d_0\omega^2\sin(\theta - \phi)\, cos\omega t + \ gcos\theta \tag{6}$$

This is the final version of the Lagrangian.

4.2.3 The Euler-Lagrange Equation

The next step is to use the Euler-Lagrange Equation. This equation was formulated in the 1750's by Euler and Joseph Lagrange. Fascinatingly, the solutions to this differential equation will yield the functions for which a system is stationary. From here, stability of these stationary points can be analyzed. The Euler Lagrange Equation can be presented as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta} \qquad (7.1\ \&\ 7.2)$$

We can break this into three steps in order to show the mathematics behind this process. First, we will find the partial derivative of (6) with respect to θ.

$$\frac{\partial L}{\partial \theta} = d_0\omega^2 \cos(\omega t)\, cos(\theta - \phi) - gsin\theta \tag{8}$$

Now we will take the partial derivative of (6) with respect to $\dot{\theta}$, and then the time derivative of that.

$$\frac{\partial L}{\partial \dot{\theta}} = l \qquad \text{and} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = l\ddot{\Theta} + d_0\omega \tag{9.1 \& 9.2}$$

Now we can plug (8) and (9.2) into the Euler-Lagrange Equation (7.2).

$$l\ddot{\theta} = \frac{d_0\omega^2}{l}\cos(\omega t)\cos(\theta - \phi) \tag{10}$$

Equation (10) can be presented as follows.

$$\ddot{\theta} - \left[\frac{d_0\omega^2}{l} cos(\omega t)\cos(\theta - \phi)\right] + \frac{g}{l} sin\theta = 0 \tag{11}$$

This is our final form of the Euler-Lagrange Differential Equation. By solving this Differential Equation, we can find stationary points for the Pendulum with a Vibrating Base, and we can analyze these stationary points for stability. This process will be examined in the following sections.

4.3 Special Cases: Horizontal and Vertical Angles

Two "special cases" of the arbitrary angle are the horizontal and vertical angles. Although to constraint the length of our report we chose to only show the derivation of equations belonging to the arbitrary angle, in both the vertical and horizontal case we can use the derived equations for the arbitrary angle to derive the equations for the vertical and horizontal case by changing our value of ϕ. In the horizontal case, ϕ is $0°$ and in the vertical case ϕ is $90°$. One thing to also note in the vertical case is that the phase shift by $90°$ in equation (6), $\sin(\theta - 90°)$ can also be written as $\cos(\theta)$ by properties of sinusoidal waves. The same applications can be done to equation (11). Applying these special cases of ϕ gives us equations that are consistent with what we derived when looking at the two special cases individually.

5. AVERAGING METHOD AND EFFECTIVE POTENTIAL

From physics, it is understood that systems always tend to move towards positions of minimum potential energy. In a normal pendulum, we use this knowledge by analyzing minimum values of potential energy to determine stability points of a pendulum.

Although, as previously mentioned, the inverted pendulum with a vibrating base has similar characteristics as a normal pendulum with smooth sinusoidal motion, the small oscillations caused by the vibrating base disrupt the smooth motion of the pendulum throughout each period and cause difficulty in analyzing stabilities. In order to deal with this complication we employ averaging techniques where we take an average over the period of rapid oscillation in order to treat motion as a single, smooth function. This helps us derive what is called an effective potential.

The effective potential is by definition, a mathematical expression that combines two, often opposing, forces into a single potential. In our case, it combines the normal smooth forces that are applied in a sinusoidal pendulum with the fast forces that are created by the oscillating base. What this allows us to do is to employ the same techniques we use in a normal pendulum to determine stability conditions.

5.1 Calculation of Effective Potential

In order to find the effective potential we begin with our differential equation found using the Euler Lagrange Equation and the Lagrangian

$$\ddot{\theta} - \frac{d_0\omega^2}{l}\cos(\omega t)\cos(\theta - \rho) + \frac{g}{l}\sin(\theta) = 0 \tag{12}$$

Since the motion of the pendulum consists of the "smooth" motion of the swinging pendulum and the "rapid" motion of the vibrating base, we separate the variable θ into two variables

$$\theta(t) = X(t) + \xi(t) \tag{13}$$

Where X is the "smooth" motion and ξ is the small, "rapid" motion. The differential equation can then be expanded into the first order approximations:

$$\ddot{X} + \ddot{\xi} = -\frac{g}{l}\sin(X) - \xi\frac{g}{l}\cos(X) + \frac{d_0\omega^2}{l}\cos(\omega t)\cos(X - \rho) - \xi\frac{d_0\omega^2}{l}\cos(\omega t)\sin(X - \rho) \tag{14}$$

From here, we can equate the corresponding terms on the left and right sides of this equation. For the rapid motion, this is

$$\ddot{\xi} = -\xi\frac{g}{l}\cos(X) + \frac{d_0\omega^2}{l}\cos(\omega t)\cos(X - \rho) - \xi\frac{d_0\omega^2}{l}\cos(\omega t)\sin(X - \rho) \tag{15}$$

However, since we have the assumption that ξ is very small compared to ω, the first and last terms on the right side of this equation can be safely neglected due to the ω^2 in the middle term, yielding

$$\ddot{\xi} = \frac{d_0\omega^2}{l}\cos(\omega t)\cos(X - \rho) \tag{16}$$

We then solve for ξ using integration, making use of the assumption that X, the "slow" motion, remains nearly constant over the period of the rapid motion, and that all constants of integration must be zero, because the rapid motion is sinusoidal centered at zero. Thus, we have

$$\xi = -\frac{d_0\omega^2}{l}\cos(X - \rho)\iint\cos(\omega t)\,dt^2 = -\frac{d_0}{l}\cos(X - \rho)\cos(\omega t) \tag{17}$$

Returning to the separated differential equation, we can now remove $\ddot{\xi}$ and its equivalent expression found above to yield

$$\ddot{X} = -\frac{g}{l}\sin(X) - \xi\frac{g}{l}\cos(X) - \xi\frac{d_0\omega^2}{l}\cos(\omega t)\sin(X - \rho) \tag{18}$$

Now we apply the averaging technique, in which we take an average of each term over the period of the rapid movement. We can treat the slow motion as constant over this period, but the rapid movement, being sinusoidal, will average out to a constant value in the end. For now, we denote this averaging using bar notation:

$$\overline{\ddot{X} = -\frac{g}{l}\sin(X) - \xi\frac{g}{l}\cos(X) - \xi\frac{d_0\omega^2}{l}\cos(\omega t)\sin(X-\rho)} \tag{19}$$

We now substitute our solved expression for ξ in the last term and take advantage of the fact that the middle term only contains a factor of ξ and is therefore negligibly small. This yields

$$\ddot{X} = -\frac{g}{l}\sin(X) + \overline{\frac{{d_0}^2\omega^2}{l^2}\cos^2(\omega t)\cos(X-\rho)\sin(X-\rho)} \tag{20}$$

This can be factored into differential form

$$\ddot{X} = -\frac{g}{l}\sin(X) + \left(\frac{1}{\omega^2}\right)\overline{\left(\frac{d_0\omega^2}{l}\cos(\omega t)\sin(X-\rho)\right)\frac{d}{dX}\left(\frac{d_0\omega^2}{l}\cos(\omega t)\sin(X-\rho)\right)} \tag{21}$$

which can be further reduced to

$$\ddot{X} = -\frac{d}{dX}\left(-\frac{g}{l}\cos(X) - \overline{\frac{1}{2}\frac{d_0^2\omega^2}{l^2}\cos^2(\omega t)\sin^2(X-\rho)}\right) \tag{22}$$

The averaging allows us to substitute $\overline{\cos^2(\omega t)} = \frac{1}{2}$, since this is the average value of the square of cosine over its period. Disregarding the abuse of notation, for simplicity in interpreting the results, we also return to using θ instead of X, and remove the averaging bars:

$$\ddot{\theta} = -\frac{d}{d\theta}\left(-\frac{g}{l}\cos(\theta) - \frac{1}{4}\frac{{d_0}^2\omega^2}{l^2}\sin^2(\theta-\rho)\right) \tag{23}$$

We have obtained an equation in the same form as the general equation of motion for physical systems, which is

$$\ddot{x} = -\frac{dU}{dx} \tag{24}$$

Where x is the coordinate of the system and U is the potential energy of the system. Thus, the equation we have arrived at via the averaging technique can be viewed as a sort of potential energy of the system, and the expression

$$U_{eff} = -\frac{g}{l}\cos(\theta) - \frac{1}{4}\frac{{d_0}^2\omega^2}{l^2}\sin^2(\theta-\rho) \tag{25}$$

is called the "effective potential" of the system.

5.2 Significance of the Effective Potential

It is imperative to note that this is *not* the actual potential energy of the pendulum, which depends only on gravity. Rather, because of the separation of variable and the averaging technique we employed, this equation simplifies the motion of the pendulum into a single smooth motion that acts as if its potential energy were the effective potential given by the above expression. If we substitute our previously found expression for ξ, we notice that

$$U_{eff} = -\frac{g}{l}\cos(\theta) + \frac{1}{2}\bar{\dot{\xi}}^2 \tag{26}$$

Which represents the effective potential as a combination of the actual potential energy of the slow motion and the kinetic energy of the rapid motion. This gives us another way of viewing the potential energy as simply the energy that the system will have at any given point, as the vibration of the base results in a constant input of energy, as shown above, which is what creates the stability in the vertical position (for sufficient inputs of energy).

6. STABILITY ANALYSIS

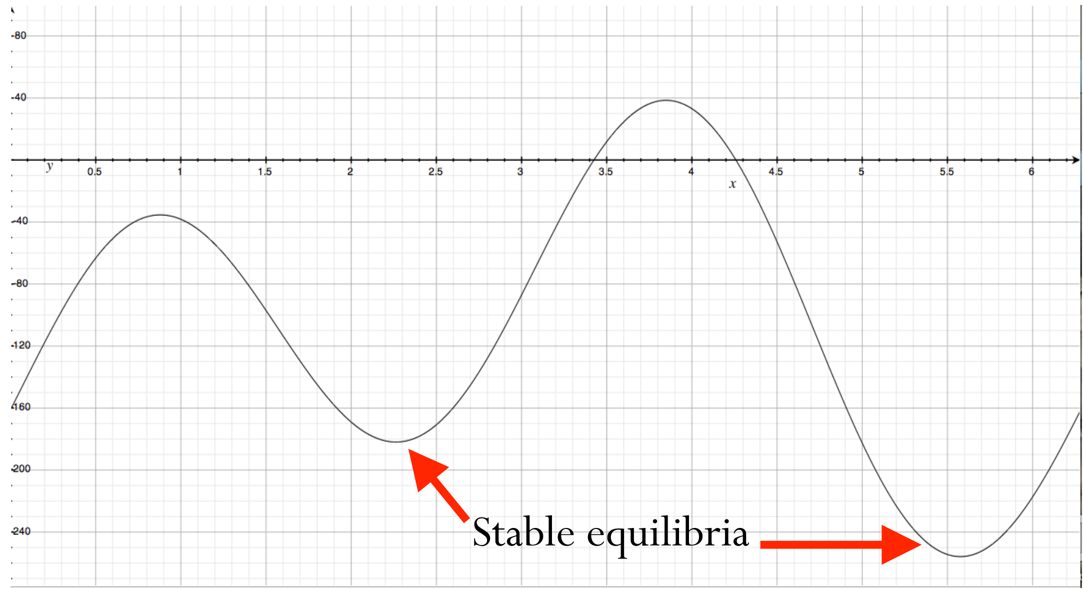
Having arrived at the "effective potential" of the pendulum's motion, we can now analyze the stability of the pendulum. Stability occurs at local minima of potential energy, including effective potential energy. In this case, we see that the effective potential

$$U_{eff} = -\frac{g}{l}\cos\theta - \frac{d_o^2\omega^2}{4l^2}\sin^2(\theta-\phi)$$

Is 2π periodic. When the frequency ω of the base vibrations is small, this function has one minimum near $\theta = 0$, which means that the pendulum will be stable only near the straight down position. This is as expected, as the straight down position is stable for a pendulum whose base is not vibrating as well. At a certain frequency, which depends on the base angle ϕ, a second stable point appears. For the horizontal or vertical cases, where $\phi = 0$ or $\phi = 90°$, respectively, this critical frequency is

$$\omega > \frac{\sqrt{2gl}}{d_0}$$

Unfortunately, the general expression for both the critical frequency and the angles of stability for an arbitrary angle ϕ is many pages long, as it involves symbolically solving a fourth order polynomial. However, since the equation for the effective potential is known, the stability points for any set of parameters can be easily determined by simply finding the minima of the above expression. For example, for a base angle of $\phi = 45°$, with parameter values $\omega = 275.62\frac{rad}{sec},\ l = 0.187\ m,\ d_0 = 0.02\ m$, the angle of stability is $\theta = 129°$, or $39°$ above the horizontal. The graph of the effective potential for this case is shown below.



For the case of a vertical base, the second stability point is always at exactly $\theta = 180°$, or straight up. In fact, there is a range of angles within which the pendulum will return to the straight up position if it is perturbed slightly. This "range of stability" is can be easily determined from the effective potential, as it is simply the range between the two maxima of the effective potential that appear at the minimum frequency. The range is given by

$$\cos^{-1}\left(\frac{-2gl}{d_0^2\omega^2}\right) \quad \leq \quad \theta_s \quad \leq \quad 2\pi - \cos^{-1}\left(\frac{-2gl}{d_0^2\omega^2}\right)$$

where the two angles shown are the maxima. Note that the minimum frequency required for a second stability point can easily be found from these expressions, as the argument for inverse cosine must be between -1 and 1.

For the horizontal case, a slightly different phenomenon is observed. The second stability point that appears at the minimum frequency is given by

$$\theta_s = \cos^{-1}\left(\frac{2gl}{d_0^2\omega^2}\right)$$

From this, it can be shown that the angle of stability begins at the straight downward position, then moves asymptotically towards the horizontal position as the frequency increases. Once again, the argument of the inverse cosine function demonstrates why the minimum frequency must be met in order for a second stable point to appear.

From these two special cases, and the equation for effective potential for an arbitrary base angle, it can be shown that an arbitrary angle of the base produces a regime somewhere in between these two. In general, the "first" stable angle is *not* in the straight down position, but slightly offset by an amount that depends on the precise angle of the base. This offset is largest when $\phi = 45°$. As the frequency passes the critical frequency, a second minima starts to appear, beginning with the larger values of ϕ. Essentially, as the frequency increases, the angle of stability approaches the angle of the base asymptotically. For the special case where the base is perfectly vertical, this asymptotic dynamic disappears, and the angle of stability becomes exactly the same as the angle of the base.

7. PHYSICAL PENDULUM

While the analysis presented thus far is very useful for determining qualitative results, particularly for the vertical case, the assumptions used are ideal and unrealistic. In particular, the model we used assumes that the pendulum has all of its mass concentrated at one point at its end. In reality, the pendulum used was closer to a rod with uniform density, although it must be noted that even this is an idealized assumption. To account for this "physical pendulum", as opposed to the simple pendulum, the moment of inertia and center of mass must be taken into account. This is done by first considering the pendulum length to be half the length of the rod, as this is the distance to the center of mass of a uniformly distributed mass. Then, the rotational kinetic energy of the rod must be added to the Lagrangian to account for the moment of inertia. Thus, the Lagrangian becomes

$$L = \frac{7}{12} l\dot{\theta}^2 + d_0\omega^2 \sin(\theta - \phi)\cos(\omega t) + g\cos\theta$$

From here, the same steps as described above are followed, as the only major difference is the coefficient. The final effective potential is now

$$U_{eff} = -\frac{6}{7}\frac{g}{l}\cos\theta - \frac{36}{49}\frac{d_0^2\omega^2}{4l^2}\sin^2(\theta - \phi)$$

This effective potential results in computational angles of stability slightly closer to the angles observed in the experiment than the simple pendulum model. However, there are still many assumptions made in this model, such as neglecting friction and air resistance, that result in experimental error that cannot be accounted for by simple error analysis.

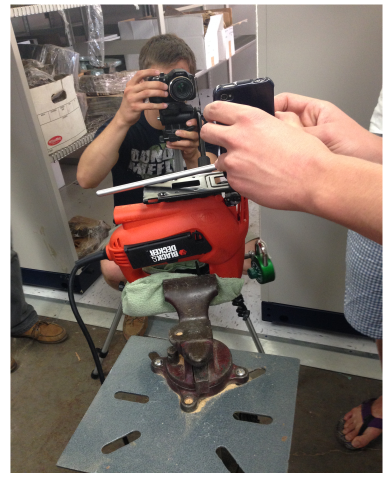
8. EXPERIMENT

8.1 EQUIPTMENT

Equipment Name	Functionality
Black and Decker JS515 Jigsaw	The Jigsaw has a metal rod to act as the pendulum
Zip Tie	To tie the Jigsaw to have a solid frequency
Clamp	Fix the pendulum at constant predetermined angles
Casio EX-FH25 High Speed Digital Camera	Absorb the motion of the vibrating pendulum
Ruler	Measure the length of the pendulum and the vibrating

	base
Micrometer	Used to measure the diameter of the pendulum
Protractor	Used to measure the angle of the base
IPhone 4	Used as the timer to calculate the frequncy

8.2 PROCEDURES



For our experiment, we shot eight videos in different angles. We have one in the horizontal angle, three in arbitrary angle, four in the vertical angle. The first two videos have a lower frequency because we shrink the zip tie after the second video. To measure the frequency, we used an iPhone timer to record the time.

8.2 RESULTS

By using fram-by-fram analysis and on-screen measurements, we are able to determine the. Our results are listed in the table below.

Raw Data Table	Measurement
Length of Pendulum (m)	.187
Diameter of Pendulum (m)	0.009525
Amplitude (m)	0.020
Minimum	0.010
Maximum	0.030
Frequency (rad/s)	275.62
Angle of Base	51°
Momentum of Inertia	$I = \frac{1}{3}ml^2$

When we compare our experimental results with our theoretical model, we have three essential models listed below.

The theoretical model is just the simple pendulum, the experimental model is the Jigsaw that we used in the lab, and the corrected theoretical model is the pendulum incorporated with moment of inertia and center of mass. Note that there is still a significant discrepancy in the table. This discrepancy is due to friction hinge and the physical pendulum is till idealized model. the corrected theoretical model doesn't change much, but it does change in the right direction.

Measurement	Theoretical	Experimental	Corrected Theoretical
Vertical Stability Angle	180°	180°	180°
Critical Angle	97°	113°	98°
Horizontal Stability Angle	83°	76°	82°
Arbitrary Stability Angle	136°	109°	135°

9. Error Analysis

Few mathematical models are a perfect description of the real word. In order to solve difficult problems, assumptions must be made, and sometimes these assumptions are less than perfect. Many assumptions were made in modelling the motion of the pendulum for example, and the averaging technique, while very useful, isn't perfect. These are some of the things that may have led to the difference in the expected theoretical results, and what was actually found through experimentation. In this section, however, we will calculate the error in our model due to measurements.

9.1 Error in Measurements

The absolute error for any measurement is defined by the unit of least count. This is the smallest measurement that can be accurately taken from the measuring device. For example, the absolute error for the length of pendulum and amplitude of base is .001 meters, because that is the smallest line on the meter stick that was used. The following table lists the errors that were considered in the model.

	Measured Value	Absolute Error	Percent Error
Length of Pendulum (l)	$l = 0.187$ meters	$\delta l = 0.001$ meters	.54%
Amplitude of Base ($d_\circ$)	$d_\circ = 0.020$ meters	$\delta d_\circ = 0.001$ meters	5.0%
Period for 60 Oscillations (T*)	T* = 1.35 seconds	δT* = 0.05 seconds	3.7%

at Angular Frequency (ω) cannot be directly measured. Instead, the variable T* is introduced and defined as the period of time required for the pendulum's base to make 60 oscillations.

9.2 Error Propagation

Since there may have been some error in three of the calculated values that are present in the relevant equations, naturally there will be some error propagation. In order to find how much error might be due to error in measurements, the Variance Formula can be utilized.

$$\delta\theta = \sqrt{\left(\frac{\partial\theta}{\partial l}\delta l\right)^2 + \left(\frac{\partial\theta}{\partial T^*}\delta T^*\right)^2 + \left(\frac{\partial\theta}{\partial d_0}\delta d_0\right)^2} \tag{27}$$

By following the process of finding stability points described in previous sections of this paper, and by making the following substitution $\omega = 2\pi * \frac{60}{T*}$, we can find the stability angle for the horizontal pendulum ($\phi = 0$) according to the following equation.

$$\theta_s(l, d_0, \omega) = \cos^{-1}\left(\frac{g}{7200\pi^2}\frac{lT^{*2}}{d_0^2}\right) \tag{28}$$

And the critical angles, or the range of stability, can be found for the vertical pendulum according to the following formula.

$$\theta_c(l, d_0, \omega) = \cos^{-1}\left(-\frac{g}{7200\pi^2}\frac{lT^{*2}}{d_0^2}\right) \tag{29}$$

Notice that the partial derivatives of (28) and (29) will be the same, except for the sign. But the Variance Formula takes the square of each of these terms anyways. This means that the Absolute Error analysis will be the same for each of these equations. If we define:

$$\alpha = -\frac{g}{7200\pi^2} = -1.38\ x\ 10^{-4} \tag{30}$$

Then we obtain the following partial derivatives (considering sign to be positive):

$$\frac{\partial\theta}{\partial l} = \frac{\alpha T^{*2}}{d_0^2\sqrt{1-\left(\frac{\alpha l T^{*2}}{d_0^2}\right)^2}} \qquad \frac{\partial\theta}{\partial T^*} = \frac{2\alpha l T^*}{d_0^2\sqrt{1-\left(\frac{\alpha l T^{*2}}{d_0^2}\right)^2}} \qquad \frac{\partial\theta}{\partial d_0} = \frac{2\alpha l T^{*2}}{d_0^3\sqrt{1-\left(\frac{\alpha l T^{*2}}{d_0^2}\right)^2}} \quad (31, 32, 33)$$

Now we can plug (31), (32) and (33) into the Variance Formula (27) along with the values in Figure 1:

$$\delta\theta(l, d_0, \omega) = \sqrt{(0.633 * 0.001)^2 + (0.176 * 0.05)^2 + (11.84 * .001)^2} \tag{34}$$

$$\delta\theta = 0.0148\ radians = .86° \tag{35}$$

This represents the amount of error that may have occurred doing to lack of accuracy in measurements. This means, the model more accurately predicts the following stability and critical angles.

Critical Angles for the Vertical Pendulum:	$\boldsymbol{\theta_c = \pm 97 \pm 0.86°}$
Stability Angle for the Horizontal Pendulum:	$\theta_s = 83 \pm 0.86\ °$

9.3 Conclusion

After analysis, it is clear that there is some error in the values that the model predicted due to possible error in the values that were plugged into the model. However, this calculated theoretical error is quite small compared to the experimental error. This means the difference between the expected and observed results must arise from elsewhere.

10. Reference

1) Landau L. D. & Lifshitz E. M., *Mechanics*, **Second Edition: Volume 1**, (Course of Theoretical Physics), (Oxford ; New York : Pergamon Press, 1976), pp 665-70. Oscillations of systems with more than one degree of freedom.

2) Mark Levi, Geometry of Kapitsa's potentials, Nonlinearity, Volume 11, 1365, 1998.

3) Acknowledgement to Dr. Ildar Gabitov for being a great mentor for our project and Three Ring Research Lab for given us the access to equipment used for the experiment.