Stability of Lagrange Points: James Webb Space Telescope

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ABSTRACT

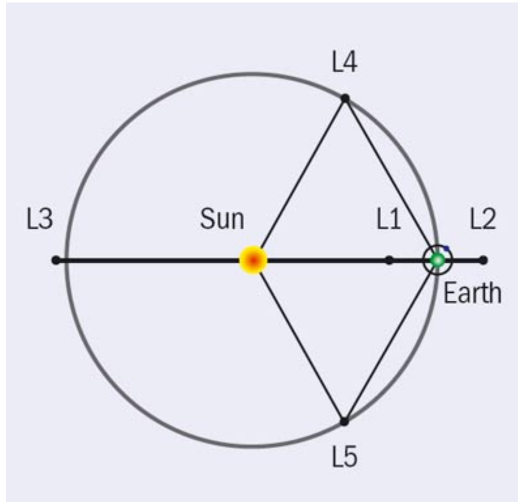
We present a detailed procedure for analyzing the stability of Lagrange points both analytically and computationally. We intend to verify our analytically derived solution with a 3D N-body code which can also be used to test the long term stability of the problem. Our problem can be applied to the next generation space telescope, the James Webb, since it will be placed at the Earth-Sun L2 point. In our final report we will provide a comprehensive solution set which includes the stability of Earth-Sun Lagrange points and analysis of the long term evolution of James Webb in its orbit.

I. INTRODUCTION

Astronomy has been as a subject studied by humans for a few thousands of years. One of the most fascinating part of astronomy is the study of the retrograde motion of the planets in the sky. Among the earliest recorded studies of the motion of planets was by Ptolemy around the year 150 CE (Jones). Ptolemy's geocentric model of the universe was accepted for many centuries following his death. This model placed the Earth at the center of the universe with the sun and planets rotating around the Earth in epicycles. It was not until Copernicus came along (1473-1543) that the idea of a heliocentric system came along. He believed that the size of the planets' orbits was proportional to the distance from the sun (Rabin 2004). This model, which features the sun in the center and the planets rotating around it, was not accepted initially, but became the accepted belief of the universe.

It was not long until the telescope was invented and people started to truly record the motion of the planets. Tycho Brahe was one of the first to build an observatory to study the planets. He made very detailed observations of the planets and was able to predict their motions with an accuracy never seen before (Van Helden 1995). His results were published after his death and were used by Johannes Kepler, his student, to create the laws of planetary motion, which make use of elliptical orbits. To this day, this is the accepted idea of how planets orbit our sun. These motions, although predicted by Kepler's Laws, can be predicted by Newton's Laws of Gravity. These explain how orbits can be stable.

The orbits are not the only places where objects can exist. Joseph-Louis Lagrange was the first to publish his theories of how three bodies can interact in a system (Lagrange 1772). Although the three body problem is not solvable analytically, Lagrange was able to determine points where all the forces balance out. An object placed at these "Lagrange points" feels no force. These are the points we will be studying in our paper.



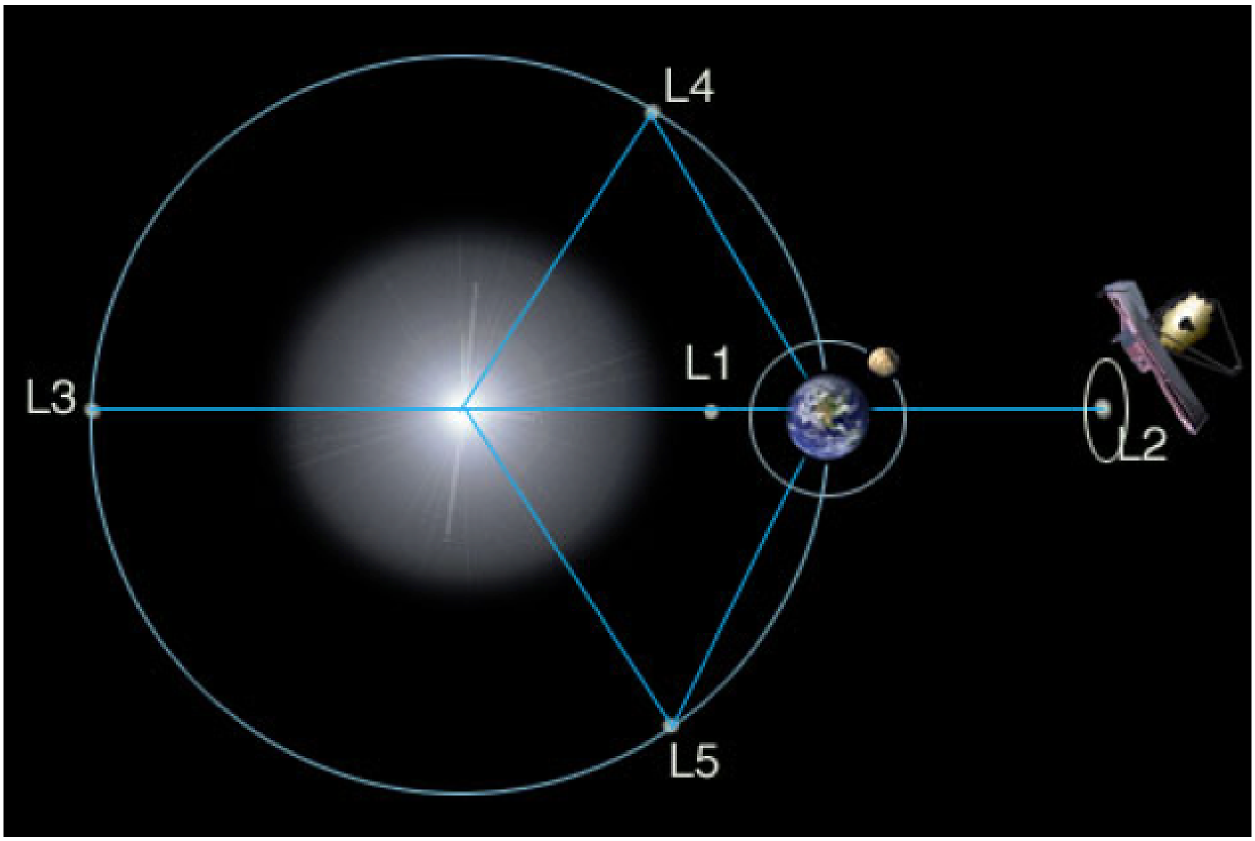
http://www.labspaces.net/blog/895/The_Lagrange_points_____Nature___s_parking_spots

There are five points in which all the forces balance out between two large bodies. For the purpose of this project, the Lagrange points considered will be those caused by the Sun and Earth. The Lagrange points all occur on the same 2D plane. The first three are in a straight line that passes through the Sun and Earth. The first point, labeled L1, occurs between the two bodies, L2 occurs just beyond the Earth. Both points are equidistant from the Earth. L3 is found outside Earth's orbital path beyond the Sun. The last two points are located in the Earth's path via equilateral triangles. L4 is where the Earth is rotating to and L5 is found trailing behind. All of these points are in constant rotation along with the Earth. The points are able to be viewed as they are in the diagram above by adopting a co-rotating frame. This frame holds the Sun and Earth fixed allowing for an easier visualization of the problem.

Our study of Lagrange points can be applied to the James Webb Space Telescope (JWST) since it will be placed at the Earth-Sun L2 point. JWST is set to launch in 2018 and will be the successor to the Hubble Space Telescope and Spitzer Space Telescope. Since 1990, Hubble has been capturing light from the UV, visible, and near-IR, covering wavelengths from 0.1 to 1.7 microns. JWST will aim to improve previous work by acquiring more precise measurements in the IR spanning 0.6 to 29 microns.

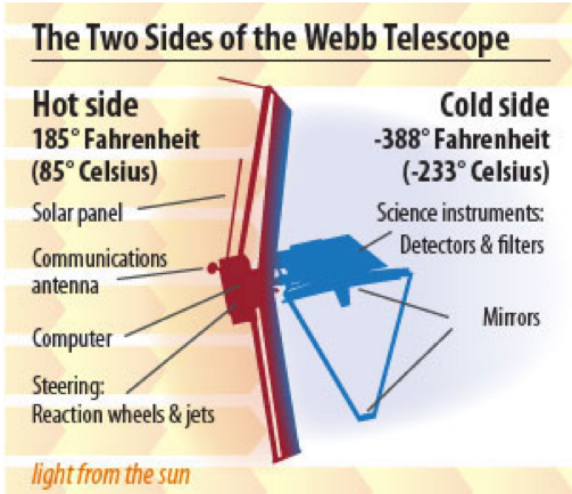
As light from stars and galaxies travels through space, its wavelength increases. Therefore, by capturing light with longer-wavelengths, JWST will be able to see objects that are farther away from Earth and from farther back in the history of the universe. Since it will be able to capture images from farther back in time, the goal of the JWST will be to investigate the formation of stars and galaxies, as well as what happened after the formation of the universe. Specifically, it has four main science themes. The theme "First Light" will be investigating the formation of the first bright objects in the universe. "Assembly of Galaxies" will explore how galaxies and dark matter formed. The other two science themes are "The Birth of Stars and Protoplanetary Systems," and "Planetary Systems and the Origins of Life." If the next "Earth 2.0" hasn't been found before JWST is launched, it will surely find it. Having a greater sensitivity at IR wavelengths than Hubble and Spitzer will allow the JWST to measure the transmission and emission spectra of planets more precisely. Measuring the amount of light at specific wavelengths that gets absorbed by a planet's atmosphere as it passes in front of a star in its line of sight can give insight on the chemical composition of the atmosphere. By specifically targeting IR wavelengths the JWST will be able to detect the presence of H20, CH4, CO and CO2 in the atmospheres of planets outside the solar system.

Lagrange points are ideal places for telescopes like JWST as the gravitational forces from the Earth and Sun cancel, leaving no net gravitational force minimizing the energy required to maintain a telescope's position.



The James Webb Space Telescope will orbit the L2 Lagrange point. (*jwst.nasa.gov*)

Currently, the Hubble Space Telescope orbits the Earth at a distance of about 570 kilometers. JWST, however, will orbit the L2 Lagrange point, about 1.5 million kilometers away, as shown in the figure above. As the Earth orbits the Sun, JWST will orbit with it, maintaining a constant position relative to the Sun and Earth. In addition to stability, this position has several advantages for an IR telescope. Its constant relative position means the Earth and Moon will never obscure the telescope's view. It is important for an IR telescope to stay cool (below 50 K on the instrument side of the telescope, as shown in the figure below), so the JWST makes use of a solar shield, essentially a large sheet of reflective material. Since the light from the Sun will come only from one direction, the solar shield can effectively block it.



At the L2 Lagrange point, the solar shield can block the light and heat from the Sun. (*jwst.nasa.gov*)

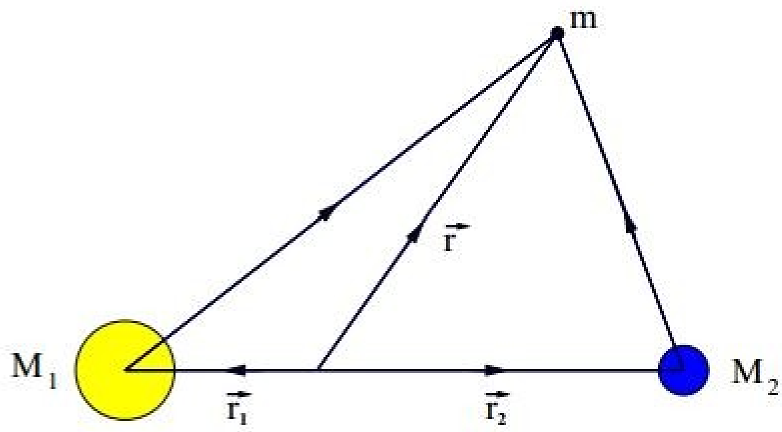
This project will further investigate the stability of all the Lagrange points and JWST's orbit around L2.

In the second section, the problem we are facing in the mathematical formulation of this problem will be discussed. The third section will provide an explanation on how the code will be set up for this problem and which method is being used to achieve this code. The fourth section will discuss plans on what to do next with this problem.

II. MATHEMATICAL FORMULATION

There are many forces to consider in this problem, including gravity, the coriolis force, and the centrifugal force. "Newton's Universal Law of Gravitation states that any two objects exert a gravitational force of attraction on each other" that is proportional to the mass of the objects and inversely proportional to the square of the distances ("Newton's Law of Gravitation" 1997). This is what causes planets to orbit stars, masses to accrete and form planets, and the stars to condense from gas. The other two forces are important when considering a co-rotating frame. The centrifugal force is the fictitious force that draws an object away from the center of rotation. Because of inertia, objects want to keep moving in a straight path until a force is applied to it; the centrifugal force is the effect of inertia of such objects. The coriolis force is a fictitious force that acts in a rotating frame of reference and gives objects a curved path even if in a non-rotating frame, such a path is straight.

We begin with Newton's Law of Gravitation in a stationary reference frame:



$$\vec{F}_{stationary}=\frac{-GM_1m}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1)-\frac{GM_2m}{|\vec{r}-\vec{r}_2|^3}(\vec{r}-\vec{r}_2)$$

The equation above describes the net magnitude and direction of force that the James Webb Telescope will feel from the earth and the sun system.. Our goal is to seek solutions to $\vec{F}=0$ that maintain constant separation between the three bodies. The positions of the three bodies that satisfy this requirement will be our Lagrange points.

The easiest way to find our solutions is to adopt a co-rotating reference frame in which the earth and sun are separated by a constant distance R, and the frame is rotating about the sun/earth system's center of mass with an angular velocity Ω given by Kepler's law:

$$\Omega=\sqrt{\frac{G(M_1+M_2)}{R^3}}$$

The effective force in this frame requires the addition of the coriolis and centrifugal force, but this reference frame has the great advantage that relative positions between bodies will be independent of time (assuming that the James Webb Telescope is much less massive than the sun and earth). Our new equation for force is thus:

$$\vec{F}_{\Omega}=\vec{F}_{stationary}-2\mathrm{m}\left(\vec{\Omega}\,X\,\frac{d\vec{r}}{dt}\right)-m\vec{\Omega}\,X\,(\vec{\Omega}\,X\,\vec{r})$$

Since we are solving for a static situation we set $\frac{d\vec{r}}{dt}=\vec{F}=0$ and solve for $\vec{r}$. To simplify this process we make the following substitutions:

$$a=\frac{M_2}{M_1+M_2}, \beta=\frac{M_1}{M_1+M_2}, x=R(u+\beta)$$

Such that:

$$\vec{r}_1=-aR\hat{i}, \vec{r}_2=\beta R\hat{i}$$

Here we have introduced the variable u to represent the distance from the earth to a given point along the x axis. Physical intuition tells us that we should first look at situations in which the stable point lies along the x-axis. With y set to 0 and m set to one (which can be done due to the fact we have set $F=0$) we reduce our task to solving three fifth-order equations given by:

$$\left(s_0+2s_0u+\left(1+s_0-s_1\right)u^2+2u^3+u^4\right)=0$$

$$s_0=sign(u)\, s_1=sign(u+1)$$

This equation cannot be solved for all *a*, but if we approximate that the mass of the earth is much less than the mass of the sun (*a*<<1) then we can find our solutions in closed form. The three possible cases we need to consider correspond to ($s_0, s_1, s_2=(-1,1),(1,1),(-1,-1)$ and these correspond to our first three Lagrange points (L1, L2, and L3):

$$L1: \quad \left(R\left[1-\left(\frac{\alpha}{3}\right)^{1/3}\right],0\right),$$

$$L2: \quad \left(R\left[1+\left(\frac{\alpha}{3}\right)^{1/3}\right],0\right),$$

$$L3: \quad \left(-R\left[1+\frac{5}{12}\alpha\right],0\right).$$

Plugging in the values for the mass of the earth and the sun to find alpha and the distance between the earth and sun for R we find that the first and second points correspond to locations about 1.5 kilometers on either side of the earth (along the x-axis) and the third location is slightly farther away from the sun than the earth is, but on the complete opposite side of the sun.

To solve for the remaining Lagrange points, we must look to solutions in which the y-component is non-zero. While this problem is slightly more tedious, the only difference is that we break $\vec{F}_{\Omega}$ down into parallel and perpendicular components and set each force component equal to zero. The solutions are as follows:

$$L4: \quad \left(\frac{R}{2}\left(\frac{M_1 - M_2}{M_1 + M_2}\right), \frac{\sqrt{3}}{2}R\right),$$

$$L5: \quad \left(\frac{R}{2}\left(\frac{M_1 - M_2}{M_1 + M_2}\right), -\frac{\sqrt{3}}{2}R\right).$$

The fourth and fifth Lagrange points are held static due to the balance between the centrifugal force and the gravitational forces of the earth and sun. These two points are symmetric about the x-axis and at a distance R from each mass, thus forming equilateral triangles with the sun, the earth, and the James Webb Telescope each at a vertex.

In future work, we will perform linear stability analysis about each of these points in order to determine whether or not the James Webb Space Telescope will be able to maintain its position with minimal correction. While often times it is easier to address stability by looking at the potential energy of the system, in this case our potential energy depends on velocity (due to the coriolis force) and this method will not work. Thus we will linearize the equation of motion and examine the effects of small perturbations from equilibrium.

III. COMPUTATIONAL FORMULATION

In order to verify our analytical solution we are going to create a 3D N-body code to assess the stability region of our solution as well as investigate the long term evolution of a body in orbit. Creating a code to assess our solution will have it own set of caveats associated with the numerical schemes in place however we aim to combat them by implementing high order expansions to help preserve our constants of motion. Our code will be written using a combination of Python and C/C++. We will do all of the plotting and setting up of initial conditions in Python and outsource the computational work to a better suited platform, C++.

To conserve energy and accuracy in our code we will invoke a fourth order method for updating the position and velocity. The fourth order method is a *predictor-corrector* method which uses a lower order explicit method to obtain our predicted values and then an implicit method in our corrector step. The method we will use was created by Charles Hermite and is a higher order method of the leapfrog integration technique. To obtain our fourth order expressions for the position and velocity, first write down the Taylor series expansions for the position and its first three derivatives:

$$x_{i+1} = x_i + \Delta t v_i + \tfrac{\Delta t^2}{2} a_i + \tfrac{\Delta t^3}{6} j_i + \tfrac{\Delta t^4}{24} s_i + O(\Delta t^5)$$
$$v_{i+1} = v_i + \Delta t a_i + \tfrac{\Delta t^2}{2} j_i + \tfrac{\Delta t^3}{6} s_i + \tfrac{\Delta t^4}{24} c_i + O(\Delta t^5)$$
$$a_{i+1} = a_i + \Delta t j_i + \tfrac{\Delta t^2}{2} s_i + \tfrac{\Delta t^3}{6} c_i + O(\Delta t^4)$$
$$j_{i+1} = j_i + \Delta t s_i + \tfrac{\Delta t^2}{2} c_i + O(\Delta t^3)$$

Here j_i is the third derivative of the position, known as the “jerk,” s_i is the fourth derivative, the “snap” and c_i is the fifth known as the “crackle” and as you can guess the sixth derivative is the “pop.” If we solve the last two expressions for the snap and crackle terms, we get:

$$\Delta t^2 s_i = 6(a_{i+1} - a_i) - 2\Delta t j_{i+1} - 4\Delta t j_i$$
$$\Delta t^3 c_i = -12(a_{i+1} - a_i) + 6\Delta t (j_{i+1} + j_i)$$

We can know substitute these into the Taylor series of the velocity and then perform a few more manipulations to obtain the position and we have:

$$v_{i+1} = v_i + \tfrac{\Delta t}{2}(a_{i+1} + a_i) - \tfrac{\Delta t^2}{12}(j_{i+1} - j_i) + O(\Delta t^5)$$
$$x_{i+1} = x_i + \tfrac{\Delta t}{2}(v_{i+1} + v_i) - \tfrac{\Delta t^2}{12}(a_{i+1} - a_i) + O(\Delta t^5)$$

There is only one issue with this so far and that is the a_{i+1} depends on x_{i+1} which has not been calculated. We can solve this issue by implementing a predictor corrector method. We will start by predicting the approximate new positions and velocities using less-accurate expressions:

$$x^p{}_{i+1} = x_i + \Delta t v_i + \tfrac{\Delta t^2}{2} a_i + \tfrac{\Delta t^3}{6} j_i$$

$$v^p{}_{i+1} = v_i + \Delta t a_i + \tfrac{\Delta t^2}{2} j_i$$

We left out the terms with $O(\Delta t^4)$ in the position and $O(\Delta t^3)$ in the velocities since we can solve for the acceleration and jerk directly. For N-body problems we can compute the acceleration and jerk as follows: (here the *i* means particle *i* not timestep *i*!)

$$a_i = -G \sum_{j=1 \& j \neq i}^{N} \frac{m_j r_{ij}}{r^2{}_{ij}}$$

Now differentiate the acceleration with respect to time to obtain the jerk:

$$j_i = -G \sum_{j=1 \& j \neq i}^{N} m_j \left(\frac{v_{ij}}{r^3{}_{ij}} - 3\frac{(r_{ij} \cdot v_{ij}) r_{ij}}{r^5{}_{ij}} \right)$$

$$r_{ij} = r_i - r_j \text{ and } v_{ij} = v_i - v_j$$

Using the four equations directly above we can put it all together and get:

$$v_{i+1} = v_i + \tfrac{\Delta t}{2}(a^p{}_{i+1} + a_i) - \tfrac{\Delta t^2}{12}(j^p{}_{i+1} - j_i) + O(\Delta t^5)$$

$$x_{i+1} = x_i + \tfrac{\Delta t}{2}(v^p{}_{i+1} + v_i) - \tfrac{\Delta t^2}{12}(a^p{}_{i+1} - a_i) + O(\Delta t^5)$$

Now that we have a method in place to evaluate motion in our N-body code we need to setup the initial conditions. To simplify our code slightly we will impose a perfectly circular orbit for the Earth by giving it an initial velocity equal to the circumference of its orbit divided by its period. To assess the stability of our analytically derived solution we will impose a co-rotating frame such that the Sun and Earth are stationary. This can easily be done by correcting the positions of all our bodies using the rotation matrix.

After we validate the physics for just the Sun and Earth system by measuring the conservation of energy as the Earth orbits the Sun over a long period of time we will test our analytically derived Lagrange points. We intend to perform an in depth analysis of the L2 point since it directly pertains to the JWST mission. We can computationally create a phase diagram of the region which will give us insight to the size of the stability region. Additionally it would be interesting to see if the Moon could throw the JWST out of a stable orbit or if the presence of a second telescope at the L2 point would have any consequences on the stability of both telescopes.

IV. CONCLUSION

In our next paper, will will solve for the stability of each Lagrange point analytically. We will also test the stability of the Lagrange points in our code. With our code we will be able to investigate how long the Lagrange points will be stable, and what would happen if we perturb objects from the L2, including figuring out its orbit, and whether it can remain close to the original position.

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