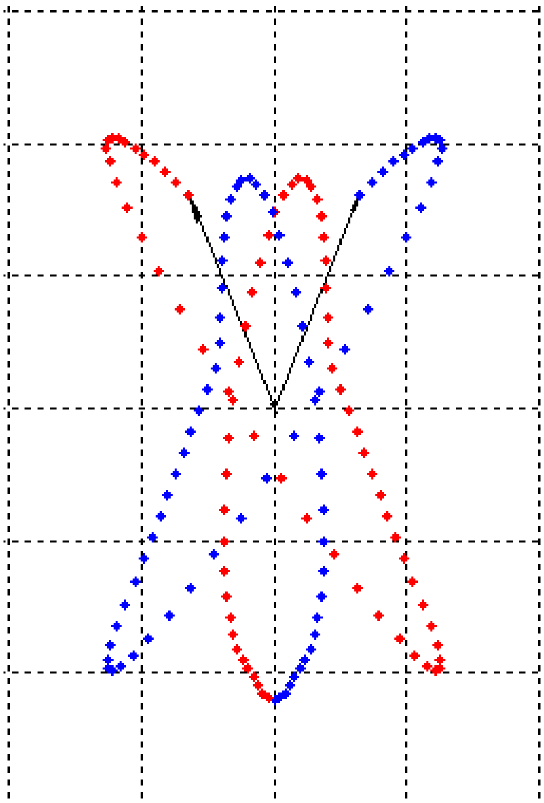

THE DYNAMICS OF THE ELASTIC PENDULUM

A group project proposal with preliminary research
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MATH MODELING MIDTERM REPORT

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1. An Introduction to the System Considered

The system of the elastic pendulum consists of a spring, connected to a pivot, suspending a mass. This spring can have many different properties among these are stiffness which can be considered a constant in some practical cases, so the spring has a linear reaction force when extended and compressed. Typically a spring that one would find in real-life applications is either an extension spring *or* a compression spring (and this will be discussed in greater detail in section 5) however for simplified models and preliminary research we find it practical to consider the case where this spring behaves in a Hookian manor in both extension and compression. This assumption, however, does beg this question among others: Does the spring bend as shown in Figure 1 when compressed ever, and how would this effect the behavior of the system? One way to answer this question is to acknowledge that assumptions such as the ones that follow need to be made to analyze this system, and in any case, behaviors such as these are not easy to analyze and would involve making many more assumptions that could be equally unsatisfying. We will be considering two regimes of this system in our preliminary research with the same assumptions and will pose questions for further research with suggestions for different assumptions. The current assumptions are:

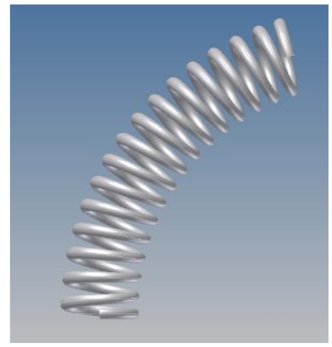


Figure 1

1. The spring is massless, cannot bend, and has a reaction force when stretched described by Hooke's law.
2. The mass is a point mass at the free end of the spring.
3. There is no friction in the system to be considered.
4. The spring and the pivot do not *prohibit* any motion of the mass –that is they essentially do not exist other than to exert the spring force on the point mass.

The system we have described looks like Figure 2 if both the spring and the mass can move through surface that the pivot is mounted on. At this point the system is starting to look like a spring mass system –and it also looks a lot like a simple pendulum. So we will start to derive the equations of motion of each of these components and see how far this gets us.

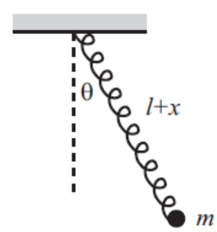


Figure 2

2. An Introduction to the Equations of Motion

The problem of the dynamics of the elastic pendulum can be thought of as the combination of two other solvable systems: the elastic problem (simple harmonic motion of a spring) and the simple pendulum.

Take simple harmonic motion of a spring with a constant spring-constant k having an object of mass m attached to the end. When the mass is "pulled" on, displacing the spring from its equilibrium position, Hooke's Law comes into play. This causes the spring to exert an elastic force to restore the spring to equilibrium.

To begin solving the system, combine Newton's second law and Hooke's law:

$$F = m\frac{d^2x}{dt^2} = -kx$$

We find that from here we have a second order differential equation for the spring's displacement *x* with respect to time. This gives us a solution of a sinusoidal function:

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t) = Acos(\omega t - \varphi)$$

Where $\omega = \sqrt{k/m}, A = \sqrt{c_1^2 + c_2^2}, \varphi = \tan^{-1}(c_2/c_1)$. The velocity and acceleration of the system can also be found from here, taking the first and second derivative of the position equation:

$$v(t) = -A\omega\sin(\omega t + \varphi)$$

$$a(t) = -A\omega^2\cos(\omega t + \varphi)$$

From here we can find the frequency *f* from the angular frequency ω with $\omega = 2\pi f$ and the time period T with

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}.$$

Now, take the simple pendulum problem. For the purpose of this paper we shall make several simple assumptions (an ideal case). This idealized mathematical model assumes that this pendulum has an object with mass attached to the end of an inelastic, massless cord. In this system there is no friction in the pendulum's pivot, no air drag, et cetera, causing this ideal pendulum to swing with a constant amplitude once given an initial push. We shall also restrict the size of the oscillation by means of the *small-angle approximation*, or assuming that the initial angle is much less than 1 radian, $\theta \ll 1$, allowing the usage of the approximation $\sin(\theta) \approx \theta$.

The small-angle approximation then yields the harmonic oscillator equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

Further approximations now must include initial conditions $\theta(0) = \theta_0$ and $\frac{d\theta}{dt}(0) = 0$. The solution can be found from here (recalling that $\theta_0 \ll 1$, with θ_0 the maximum angle between the pendulum rod and the "vertical" or normal):

$$\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{l}}t\right)$$

The period of the motion, or the time for one complete oscillation is then

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Note that it is only under this special small-angle case that the time period is independent of initial amplitude angle.

While we can solve simple cases of both the elastic problem and the pendulum problem, to assume that an elastic pendulum is simply a combination of these two sets of equations is misguided. This more complicated system cannot be solved with a second-order differential equation and even forces mathematicians looking at the system into chaos theory.

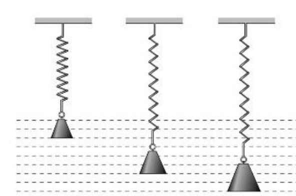
3. Deriving Equations of Motion Using the Lagrangian

We shall use Cartesian coordinates for the derivation of equations of motion even though it may seem less intuitive than spherical coordinates, because this will make it easier to express how the equation can be modeled in MATLAB. We will consider positive z to be pointing upward, and we will consider the origin to be centered at the pivot. This is the general form of the Lagrangian:

$$L = T - V$$

T is the kinetic energy of the system. The kinetic energy is equal to $\frac{1}{2}mv^2$ where m is the mass, and v is the velocity of the mass. There is no surprise yet, and it comes as no surprise either that $=$ $\left(v_x^2 + v_y^2 + v_z^2\right)^{\frac{1}{2}}$. So letting $v_x = \dot{x}$ and so on, were the dot represents the time derivative we have the kinetic energy of the system $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ and we are now half way done.

The potential energy of the system consists of two parts: The elastic potential energy stored in the displacement of the spring from its equilibrium position, and the gravitational potential energy. Since the Lagrangian is derived for the express purpose of differentiating it we will not be concerned with the details of defining the gravitational potential energy, namely the discrepancies regarding the reference for origin and how that effects the values of the potential energy.



Starting with the elastic potential energy, we have the pleasure of the simplicity of Hooke's Law $F = -kx$ which we can integrate over a displacement to get $V_{elastic} = \frac{1}{2}kx^2$. To make this apply to our model, all we need to do is imply that there is a length that the spring with the mass attached to it, will rest at equilibrium stretched under the weight of the mass, mg, we will call this length l. Furthermore we will consider a length l_0 that is the natural, unstretched length of the spring with no weight attached. By Hooke's law the relationship between these lengths is $k(l - l_0) = mg.$ Since l_0 is a constant characteristic of the system considered, we conclude that $V_{elastic} = V_e = \frac{1}{2}k(r - l_0)^2$ where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.

Lastly, with all of the simplicity expected the gravitational potential energy of the system is $V_g = mgz.$ Things brings us to a complete Lagrangian for the system with the assumptions proposed in section 1:

$$L = \frac{1}{2}m\left(\dot{x}^2 + y^2 \dot{+} \dot{z}^2\right)^{\frac{1}{2}} - [\frac{1}{2}k(r - l_0)^2 + mgz]$$

By differentiating with respect to time with the chain rule, and equating functions of each variable to their time derivatives we get a system of ordinary differential equations:

$$\ddot{x} = -\omega_z^2 \frac{r - l_0}{r} x$$

$$\ddot{y} = -\omega_z^2 \frac{r - l_0}{r} y$$

$$\ddot{z} = -\omega_z^2 \frac{r - l_0}{r} z - g$$

Where $\omega_z^2 = \frac{k}{m}$ is referred to with subscript of z on the character ω to indicate that it does not represent a frequency of the system, but only the frequency of the spring-mass system with only vertical oscillations.

The system derived above has three degrees of freedom in space, but only two constants of motion, the total energy, $E = T + V$, in the system and the angular momentum per unit mass about the vertical represented: $h = x\dot{y} - y\dot{x}$; this accounts for two invariant forms of the system. Therefore we say that the system is not integrable in general, though in some regimes, particularly those of small oscillations, "approximate analytical solutions can be found" (Lynch, 2002)[1]. These have however already been investigated and therefore will not be a focus of our project.

Though the system in general is not integrable, there is a way to analyze the behavior of the system. This way is through numerical integration, which can be done so long as chaotic regimes (with extreme initial conditions of high amplitude) are avoided. One way of numerically integrating is through the Runge-Kutta method, which can be conveniently employed using MATLAB. ODE45 is MATLAB function which uses a variation of the fourth-order Runge-Kuttta method to model ordinary differential equations such as this one, results of this modeling will be discussed in section 4.

4. Trivial Cases and Validity of Models

Now, we consider trivial cases of the system with certain initial conditions. The behavior of an elastic pendulum can be divided into two different kinds of motions driven by two different potentials: One is swinging motion caused by gravitational potential and the other is the vertical oscillation lead by elastic potential energy. With diverse initial conditions, the system can develop diverse behavior. Our plans to investigate some of the behaviors will be outlined in section 5. Before discussing the trivial cases, we will point out what parameters and initial conditions are necessary to define in the model:

Parameters

$g \rightarrow$	The gravitational constant acting on the mass
$\omega_z^2 = \frac{k}{m} \rightarrow$	The stiffness to mass ratio
$l \rightarrow$	The equilibrium length of the spring
$l_0 \rightarrow$	The relaxed length of the spring

Initial Conditions

x_0	$\dot{x}_0$
y_0	$\dot{y}_0$
z_0	$\dot{z}_0$

When referenced, the initial conditions will be addressed in vector notation for convenience as follows:

$$\mathbf{x_0} = (x_0, \dot{x}_0, y_0, \dot{y}_0, z_0, \dot{z}_0)$$

One obvious trivial case is when the initial conditions are all null with the exception of z_0 and $\dot{z}_0$; any initial conditions of the form $(0,0,0,0,z_0,\dot{z}_0)$ will yield only vertical oscillations limited to the motions of a classical spring-mass system. A more specific trivial case has the initial condition: $(0,0,0,0,-l,0)$. In this case there is no motion whatsoever because the mass will stay at the equilibrium for forever. A more interesting case is the initial condition where the mass is balanced above the pivot, $(0,0,0,0,2l_0 - l, 0)$ where $l < 2l_0$, is an equilibrium of the system, though it is unstable (lynch, 2002)[1]. Intuitively considering the dynamics around this equilibrium point in the xz-plane shows that this point is a saddle point with the stable manifold extending down to the pivot and upward to the point that the spring breaks. The unstable manifold is horizontal axis extending from the point.

After considering these case it might make sense to consider a trivial case in which the system behaves like a pendulum, but not like a spring. To have any hope of realizing a system like this, we would obviously have to assume that the initial extension of the spring is null, and that the initial velocity in the direction normal to the spring is null. This seems promising, however in order to achieve the motion of a simple pendulum, there has to be a velocity angular velocity $\omega(\theta)$ which would follow a circular path in the case of a simple pendulum due to the tension force in the string. In the case of the elastic pendulum, the spring extends and does not follow a circular path, furthermore the equilibrium length of the spring does not exactly exist when in motion, and would otherwise shift in a manor dependent on the angle θ of the spring with the z −axis along which gravity is acting, shown in Figure 3. Thus the potential for the system to behave like a classic pendulum is destroyed. In fact, realizing any type of periodic motion is unlikely this system, the possibility for quasi-periodic motion is further discussed in section 5.

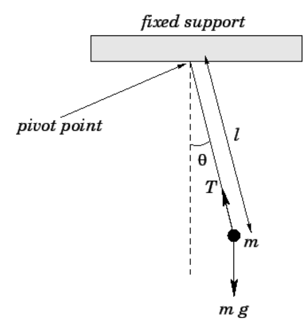


Figure 3; a simple pendulum

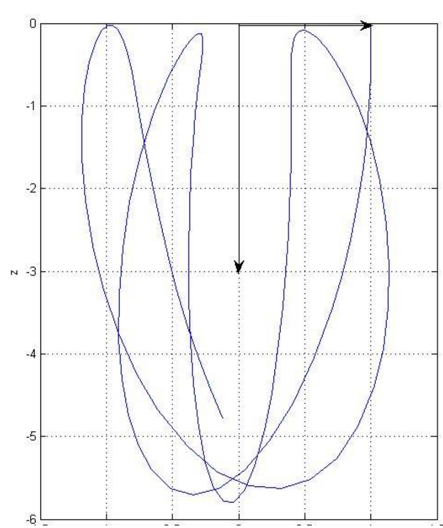
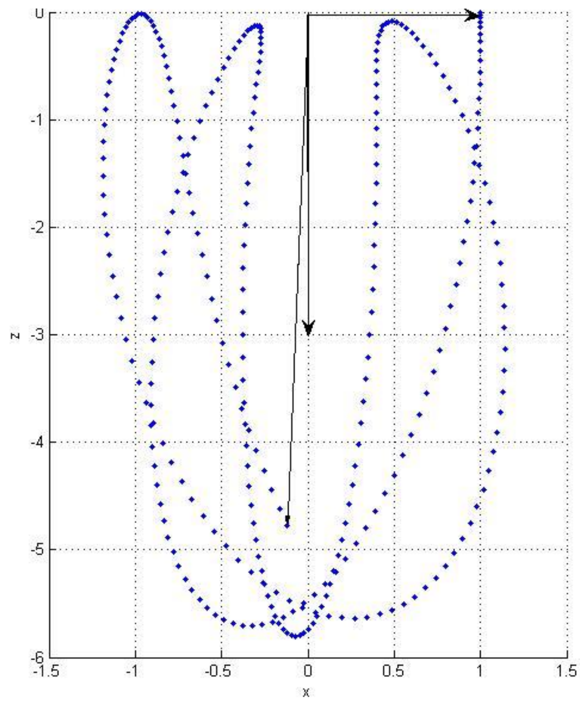


Figure 4; path of pendulum starting with spring relaxed

Figure 4 shows the pendulum starting with initial condition $(l_0, 0,0,0,0,0)$ and swinging for 10 seconds where $g = 10, w_z^2 = 5$ and $l_0 = 1$. The horizontal vector represents the initial position of the system and the vertical arrow represents the equilibrium length of the spring $l = \frac{g}{w_z^2} + l_0$.

Figure 5, alternate representation of figure 4, 30 dots per second



Here is another representation of the same model, with an extra arrow representing the position of the spring at the time of the end of the simulation (10 seconds). In this case, we have forced ODE45 to take a specific step size of 30 steps per second and we have told the model to drop a point at every step. Now we can graphically see, with the density of the points, the velocity of the spring at that position. We can also see, that over the period of 10 seconds, forcing ODE45 to take an arbitrary step size had very little effect on its representation of the motion of the spring. If we had done this without choosing the step size, the program would have chosen a step size that optimized accuracy and computation time as it did for the image in Figure 4.

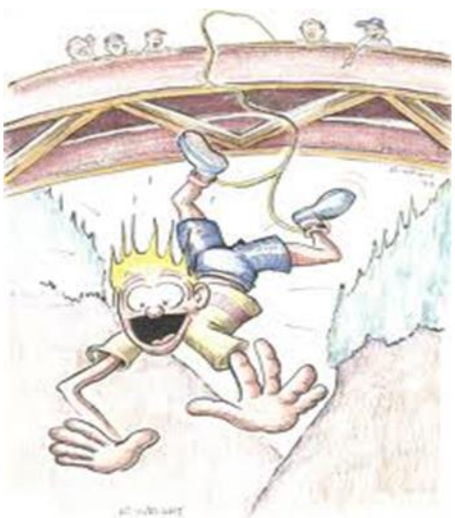
The feature to either side of the pivot in figures 4 and 5 shows the spring becoming compressed as the mass speeds toward the pivot, this is a prime example of why regimes of small amplitudes are considered for analysis of more regular motion and periodic behavior. Features like this increase error with the comparison of a mathematical model to an experimental test, because this model does not accurately describe what would happen in this situation. Since the system started its motion with no velocity in the y-direction, the model continues with this behavior, neglecting the possibility that the spring would bend under compression and send the mass moving out of the xz-plane. Though we did assume the spring does not bend in section 1, our goal is not to test our assumptions, our goal is to study the behavior of a system in as accurate of detail as possible.

5. Goal Setting

Figure 6

Figure 7

On the note of studying the behaviors of the system, we have plans to create a physical model of the system in such a way that friction is minimal. As stated in section 1, there is no spring that will behave the way we have made our assumptions, particularly regarding the assumptions that the spring does not bend and behaves ideally under compression and extension, so we plan on testing our model with many springs connected in series that are quite free to swivel between one another. The springs should be little in mass compared to the bob to realize at least a glimpse of the analogy to the assumptions outlined in section 1. Testing the motion of this using a high speed camera and possibly also a long-exposure photograph with an LED light attached to the bob would be a great way to test the validity of our model. Also, creating a version of the program that assumes that the spring only has a force under extension and not compression should be easy. This will tell us whether our model is capable of showing the motion of the spring with some accuracy, though we should not assume similar errors to accumulate as in the case of the assumptions made in section 1. None of the errors discussed involving the bending of the spring and none of the errors involving the abnormalities near the pivot would occur. What would happen is shown in Figure 6, the spring will scrunch up when it would otherwise be in compression, and given that each spring shown in Figure 6 weighs approximately 1.8 grams, this could cause errors. One error is only circumstantial, the possibility that the springs become tangled. This is not a sincere concern of ours as this would only be cause to run another trial. An unavoidable error would be that the spring passes its equilibrium point with a negative velocity along the radial axis, scrunches and pushes some mass out to the side doing work on the bob, and even more the force of the bob pulling the springs straight again will cause the displacement of the mass to do work on bob again. Though this system is not ideal, I believe we can find a bob mass and spring system with characteristic stiffness, length and mass that will make these errors relatively small so that we can test the motion of large amplitudes at least for 10 seconds or so and hope for validity.



Bungee Jumping

A secondary reason for running tests of this type is to answer a question regarding an application of the elastic pendulum. Can the safety of extreme bungee jumping be analyzed using this model? If we determine that our model represents the motion of a small scale model of bungee jumping within reasonable accuracy, we might infer that a larger scale model could hold up. This brings to question whether the safety of bungee jumping with more interesting initial conditions than the classic initial displacement of only height could be hypothesized about, and then put to a customized test with this model alone.

Quasi-Periods in Large Amplitude

Another major question we are intent upon answering is where quasi-periods can be found in large amplitudes, and how we can influence the shape of the quasi periodic forms of the system. Studying this question should prove to be a productive endeavor given the quantitative nature of the research involved. Moreover we already have the aid of our MATLAB programs that we will continue to develop into more powerful tools for calculating such behaviors of the system.

One lead seems promising in the search for quasi-periodic behavior of large amplitudes as well as a potential thrill for bungee jumping. One is shown in Figure 8 and represents a very rough and underdeveloped idea of what quasi-periodic behavior we are searching for. It

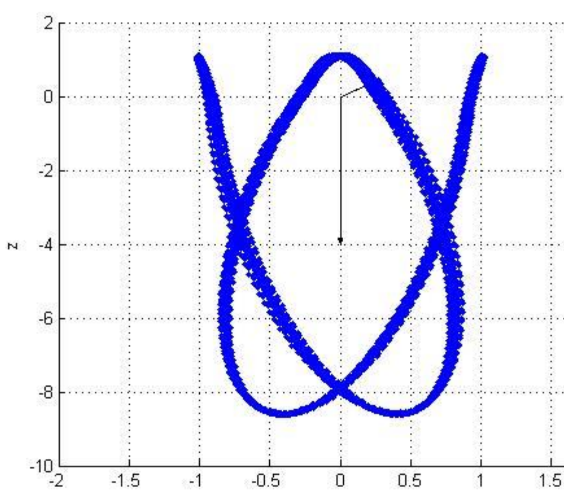


Figure 8; quasi-periodic behavior of the system, $\Delta t = 300s$, $g = 9$, $\omega_z^2 = 3$, $l = 1$ with initial conditions $(1,0,0,0,1.1,0)$

has an interesting shape, and a few features worth discussing. One is that the figure bows out in some regions, and stays tight in others, but the system shows no ambitions towards changing general shape even after minutes. The quasi-period of this motion is approximately 16 seconds. This quasi-periodic behavior is a great example of the direction we could go in for examining the logistics of testing thrilling experiences for bungee jumpers. Since the mass never enters the domain of compression of the spring, we can assume that the motion would be similar for a bungee cable. The idea of the mission would be as follows: The jumper leaves a platform at the right endpoint of the shape in figure 9 attached to a bungee cable with an initial extension, then accelerates away following the path down, then over a pivot (imagine the pivot as a pole at the vertex of the two arrows in the figure extending normal to the page) and to the other endpoint, following the same path back to the platform where Velcro is used to safely stick to a cushion. With analysis of realistic bungee cables and how they compare to spring constants of Hooke's law, and with a system in which friction is realized and experimentally accounted for, tests could lead to realizing a thrill such as this. One danger of the lure of the quasi-period must be discussed however: small miscalculations and incorrect positions could send a thrill-seeker into terror! The figure to the right shows the variance after just one "period" of the circumstance in which a jumper leaves the platform with a velocity of just $0.1 m/s$ in the y direction. This jumper indubitably misses the platform. Similar variances from an anticipated path can occur with small miscalculations in the mass of a jumper. On another note, the figure to the left shows two similar initial conditions over the course of 32 seconds. Both have the same parameters, the blue has the same initial conditions as the previous blue figures, but the cyan has $x_0 = 1.1$ instead of 1. This large divergence in path over such a short period of time shows that the behavior of the system is not very intuitive or predictable. The quasi-periods that do abound can easily be neighbored by chaotic behavior, though we do not intend to discuss the potential for chaotic behavior of the system.

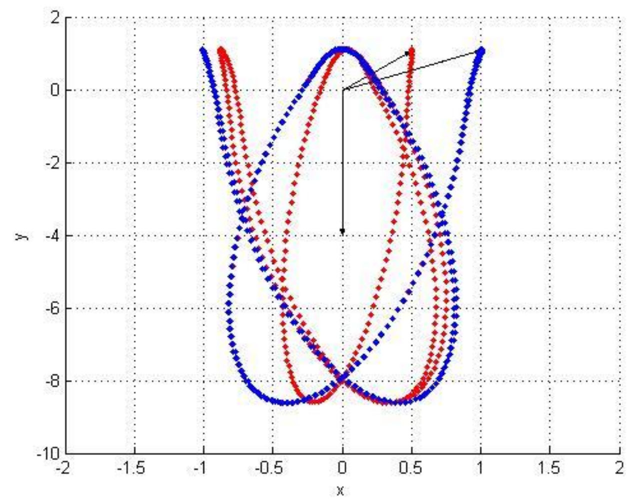


Figure 9; Blue is the same as figure 8, red has an initial velocity in y-direction of 0.1

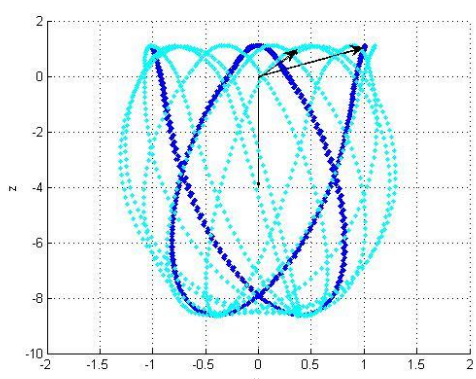


Figure 10, a small difference in initial x displacement goes a long way

If we find that we have more time for experimentation our goals for experimentation and research may expand, however we are foremost eager to see how a physical model stands up to the MATLAB models we have been referencing throughout the paper. Here we recapitulate our goals for the project:

1) Design the best physical model that we can with reasonable expenditure on resources.
2) Understand and improve our MATLAB model by comparing it to a physical model.
3) Test potential quasi-periods at large amplitudes with complex paths.

4) DISCUSS THE POTENTIAL APPLICATIONS FOR COMPREHENSION OF CERTAIN REGIMES OF THE DYNAMICS OF THE ELASTIC PENDULUM, INCLUDING EXTREME BUNGEE JUMPING.

References and Acknowledgements

- Thanks to our mentor Joseph Gibney for getting us started on the MATLAB program and the derivations of equations of motion.
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