2.5: Logarithmic Functions

**Logarithm:**
For $a > 0$ and $a \neq 1$, and $x > 0$

\[ y = \log_a(x) \text{ means } a^y = x. \]

$\log_{10} 100 = 2$ means $10^2 = 100$.
$\log_2 8 = 3$ means $2^3 = 8$.
$\log_3 1 = 0$ means $3^0 = 1$

**Example.** Evaluate:

1. $\log_2 16$

2. $\log_3 \frac{1}{81}$

**Definition.** For $a > 0$ and $a \neq 1$, the logarithmic function of base $a$ is defined as

\[ f(x) = \log_a(x) \]

for $x > 0$.

**Example.** Graph $f(x) = \log_3(x)$ and $g(x) = 3^x$. Find the domain and range of both functions.
The domain of \( f(x) = \log_a(x) \) is \( D = (0, \infty) \) and the range is \( R = (-\infty, \infty) \).

For any number \( x \), if \( f(x) = y \), then \( g(y) = x \), we say that \( f \) and \( g \) are inverse functions of each other.

\( f(x) = \log_2(x) \) and \( g(x) = 2^x \) are inverse functions.

For \( a > 0 \) and \( a \neq 1 \), \( f(x) = \log_a(x) \) and \( g(x) = a^x \) are inverse functions.

We can graph the inverse of \( f \) by reflecting the graph of \( f \) about the line \( y = x \).

**Example.** Find the domain of \( f(x) = \log_{10}(x + 3) \).
Properties of Logarithms: Let $x, y$ be an positive real numbers and let $r$ be any real number. Let $a$ be a positive real number, $a \neq 1$. Then

- 1. $\log_a xy = \log_a x + \log_a y$
- 2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
- 3. $\log_a x^r = r \log_a x$
- 4. $\log_a a = 1$
- 5. $\log_a 1 = 0$
- 6. $\log_a a^r = r$
- 7. $a^{\log_a x} = x$.

Example. Write as a common logarithm:

1. $\log_2(x + 1) + \log_2(x - 1)$

2. $2 \log_3(z + 2) - \log_3(z + 3)$

Example. Expand the logarithm

$$\log_3 \left( \frac{x^2 - 4}{xy} \right)^2$$
In order to graph a logarithmic function on your calculator for a base other than $e$ or 10, the following theorem is useful:

**The change-of-base Theorem for logarithms:**

Let $x$ be any positive real number and let $a$ and $b$ positive real numbers, $a \neq 1$, $b \neq 1$, then

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Using $\ln x$ for $\log_e x$ gives the special case:

$$\log_a x = \frac{\ln x}{\ln a}.$$

**Example.** Evaluate: $\log_7 90$

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**Solving Logarithmic Equations:**

**Example.** Solve the following Logarithmic Equations:

1. $\log_2 x = 3$
2. $\log_3(4x - 1) = 2$

3. $\log(x - 1) - \log(x + 2) = 1$

4. $\ln(x - 3) + \ln(x + 3) = \ln(x)$
Solving Exponential Equations:

**Example.** Solve the following Exponential Equations:

1. \(2^x = 3\)

2. \(e^{3x} = 2\)

3. \(3^{x+2} = 5^{2x}\)
Example. With an inflation rate of 3% per year, how long will it take for prices to double: