### 3.2: Continuity

**Continuity at** $x = c$:

A function $f$ is **continuous** at $x = c$ if the following three conditions are satisfied:

- 1. $f(c)$ is defined.
- 2. $\lim_{x \to c} f(x)$ exists, and
- 3. $\lim_{x \to c} f(x) = f(c)$.

If $f$ is not continuous at $c$, it is **discontinuous** there.

We will use this 3-step test to check if a function is continuous:

**Example.** For the following functions, draw the graph. Then determine if the functions are continuous at the indicated $x$-value:

1. 
   
   $$f(x) = x + 1 \quad \text{at} \quad x = 1$$

   **Step 1**

   **Step 2**

   **Step 3**
2. 

\[ g(x) = \frac{x^2 - 1}{x - 1} \quad \text{at} \quad x = 1 \]
3.

\[ h(x) = \frac{|x - 2|}{x - 2} \quad \text{at} \quad x = 2 \]
4.

\[ k(x) = \begin{cases} 
  x + 2 & \text{if } x \neq 3; \\
  4 & \text{if } x = 3.
\end{cases} \quad \text{at } x = 3 \]
5.

\[ l(x) = \frac{1}{x+2} \quad \text{at} \quad x = -2 \]
**Definition.** A function is **continuous** on an open interval if it is continuous at every $x$-value in the interval.

**Definition.** A function is **continuous from the right** at $x = c$ if \( \lim_{x \to c^+} f(x) = f(c) \).

**Definition.** A function is **continuous from the left** at $x = c$ if \( \lim_{x \to c^-} f(x) = f(c) \).

**Continuity on a Closed interval:**

A function $f$ is **continuous on a closed interval** $[a, b]$ if:

- 1. it is continuous on the open interval $(a, b)$.
- 2. it is continuous from the right at $x = a$, and
- 3. it is continuous from the left at $x = b$.

**Example.** The function $f(x) = \sqrt{4 - x^2}$ is continuous on the closed interval $[-2, 2]$. 
Here are the functions we have learned so far listed with the intervals in which the function is continuous:

**Polynomial function**, \( y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \), where \( a_n, a_{n-1}, \ldots, a_1, a_0 \) are real numbers:

**Continuous at**: For all \( x \)

**Rational function**, \( y = \frac{p(x)}{q(x)} \), where \( p(x) \) and \( q(x) \) are polynomials with \( q(x) \neq 0 \):

**Continuous at**: For all \( x \) where \( q(x) \neq 0 \)

**Root function**, \( y = \sqrt{ax + b} \), where \( a \) and \( b \) are real numbers with \( a \neq 0 \) and \( ax + b \geq 0 \):

**Continuous at**: For all \( x \) where \( ax + b \geq 0 \)
Exponential function, $y = P_0a^x$, where $a > 0$ and $P_0$ is the value of $y$ at $x = 0$:

Continuous at: For all $x$

Logarithmic function, $y = \log_a x$, where $a > 0$, $a \neq 1$, and $x > 0$

Continuous at: For all $x > 0$
When a function is not continuous, it has one or more points where it is discontinuous:

**Example.** Find all values $x = a$ where the following functions are discontinuous:

1. 
   \[ f(x) = \frac{x - 1}{x^2 + 2x - 3} \]

2. 
   \[ g(x) = 2^{3x-1} \]
Example. Find all values of $x$ where the piecewise function is discontinuous,

$$f(x) = \begin{cases} 
 x + 1 & \text{if } x < 1; \\
 x^2 + 1 & \text{if } 1 \leq x < 2; \\
 2x - 5 & \text{if } x \geq 2.
\end{cases}$$
Example. Find the value of the constant $k$ that makes the function continuous

$$g(x) = \begin{cases} \begin{align*} kx^2 + 1 & \text{if } x \leq 3; \\ x + k & \text{if } x > 3. \end{align*} \end{cases}$$
Example. A car rental firm charged $30 per day or portion per day to rent a car for a period of 1 to 4 days. Days 5 and 6 were then free, while the charge for days 7 through 10 was again $30 per day. Let $A(t)$ represent the average cost to rent the car for $t$ days, where $0 < t \leq 10$. Find the average cost of a rental for the following number of days:

a. 3  
b. 5  
c. 8

d. Find $\lim_{t \to 3^-} A(t)$

e. Find $\lim_{t \to 3^+} A(t)$

f. Where is $A$ discontinuous on the given interval?