3.4: Definition of the Derivative

Recall the following:

**Instantaneous rate of change:**

The **instantaneous rate of change** of \( f(x) \) with respect to \( x \) when \( x = a \) is

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

provided this limit exists.

We will give a **Geometric interpretation:**
Slope of the tangent line:

The **tangent line** of the graph of \( y = f(x) \) at the point \((a, f(a))\) is the line through this point having slope

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

provided this limit exists. If this limit does not exist, then there is no tangent at the point.

- The slope of the tangent line of the graph of \( y = f(x) \) at the point \((a, f(a))\) is the same as the instantaneous rate of change of \( f \) at \( x = a \).

- The slope of the tangent line at a point is also called the **slope of the curve** at the point.

**Example.** Let \( f(x) = x^2 + 1 \)

A. Find the slope and the equation of the secant line through the points where \( x = -2 \) and \( x = 3 \).
B. Find the slope and equation of the tangent line at \( x = -2 \).
The Derivative:

We denote

\[ f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \]

provided this limit exists.

The Derivative:

The derivative of the function \( f \) at \( x \) is defined as

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

provided this limit exists.

The function \( f'(x) \) is called the derivative of \( f \) with respect to \( x \).

The notation \( f'(x) \) is read ’\( f \)- prime of \( x \)’. 

Notice that \( f'(x) \) is a function of \( x \) as \( x \) varies.

\( f'(a) \) is the slope of the tangent line at \( x = a \) which is a number. \( f'(a) \) is the value of \( f'(x) \) evaluated at \( x = a \).

If \( f'(x) \) exists, we say that \( f \) is differentiable at \( x \). The process that gives \( f' \) is called differentiation.

Interpretations of the derivative:

1. The function \( f'(x) \) represents the instantaneous rate of change of \( y = f(x) \) with respect to \( x \). From now on we will say rate of change to mean instantaneous rate of change.

2. The function \( f'(x) \) represents the slope of the graph of \( f(x) \) at any point \( x \). If we evaluate the derivative at \( x = a \), to get \( f'(a) \), then \( f'(a) \) represents the slope of the curve or the slope of the tangent line at that point.
The difference quotient,
\[ \frac{f(x + h) - f(x)}{h} \]
represents:
- Slope of the secant line
- Average rate of change
- Average rate of change in cost, revenue, or profit
- Average velocity

The derivative,
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
represents:
- Slope of the tangent line
- Instantaneous rate of change
- Marginal cost, revenue, or profit
- Instantaneous velocity

Let \( b = x + h \) so \( h = b - x \). Then we have the following alternate form of the derivative:

The Derivative:

The derivative of the function \( f \) at \( x \) can be written as
\[ f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x} \]
provided this limit exists.
Example. Let $f(x) = \frac{2}{x}$

A. Find $f'(x)$
B. Find $f'(3)$

C. Find the equation of the tangent line at $x = 3$.

Using the point-slope form we obtain

**Equation of the Tangent line:**

The tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$ is given by the equation

$$y - f(x_1) = f'(x_1)(x - x_1)$$

provided $f'(x)$ exists.
Example. The profit, $P$, in (thousands of dollars) from the expenditure of $x$ thousand dollars on advertising is given by

$$P(x) = 1000 + 90x - x^2$$

(A) Find the marginal profit at the following expenditures: Decide in each case, whether the firm should increase the expenditure:

1. $4000$

2. $80000$
Existence of the derivative

The derivative $f'(x)$ of a function $f$ does not always exist. Here are some examples:

Example. Let

$$f(x) = |x| = \begin{cases} 
  x & \text{if } x \geq 0; \\
  -x & \text{if } x < 0.
\end{cases}$$

Does $f'(0)$ exists?
The derivative exists when a function $f$ satisfies all of the following conditions at a point:

- $f$ is continuous
- $f$ is smooth
- $f$ does not have a vertical tangent line.

The derivative does not exist when any of the following conditions are true for a function at a point:

- $f$ is discontinuous
- $f$ has a sharp corner
- $f$ has a vertical tangent line.