1.Find dy/dx for the following:

a) $x^2 - xy + y^3 = 8,$

$2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = \frac{y-2x}{3y^2-x}$.

b)$\sqrt{x+y} + \sqrt{xy} = 8,$

$\frac{1+\frac{dy}{dx}}{2\sqrt{x+y}} + \frac{y+x\frac{dy}{dx}}{2\sqrt{xy}} = 0$ or $\frac{dy}{dx}\left(\frac{1}{2\sqrt{x+y}} + \frac{x}{2\sqrt{xy}}\right) + \frac{1}{2\sqrt{x+y}} + \frac{y}{2\sqrt{xy}} = 0$ and

$$\frac{dy}{dx} = -\frac{\frac{1}{2\sqrt{x+y}} + \frac{y}{2\sqrt{xy}}}{\left(\frac{1}{2\sqrt{x+y}} + \frac{x}{2\sqrt{xy}}\right)}$$

c)$x\sqrt{1+y} + y\sqrt{1+2x} = 2x$

$\frac{x}{2\sqrt{1+y}}\frac{dy}{dx} + \sqrt{1+y} + \frac{dy}{dx}\sqrt{1+2x} + \frac{2y}{\sqrt{1+2x}} = 2$ so

$$\frac{dy}{dx} = \frac{2 - \sqrt{1+y} - \frac{2y}{\sqrt{1+2x}}}{\frac{x}{2\sqrt{1+y}} + \sqrt{1+2x}}$$

d) To find horizontal tangents, find where $\frac{dy}{dx} = 0$ and then plug the solution back into the original equation so that you can find both the x and y points. For vertical tangents, do the same, except where $\frac{dy}{dx}$ looks like $\frac{1}{0}$ (infinite slope).

2. Find the tangent line to

a) $y^2 = x^3(2-x)$ at $(1,1)$,

so the derivative can be computed as $2y\frac{dy}{dx} = 6x^2 - 4x^3$, or $\frac{dy}{dx} = \frac{3x^2-2x^3}{y}$. At $(1,1)$ the slope is 1 and hence the equation of the tangent line is $y = x$.

b) $2\left(x^2+y^2\right)^2 = 20\left(x^2+y^2\right)$ at $(3,1)$

so the derivative can be computed as $4\left(x^2+y^2\right)\left(2x + 2y\frac{dy}{dx}\right) = 40x + 40y\frac{dy}{dx}$ and hence

$$\frac{dy}{dx} = \frac{8x\left(x^2+y^2\right) - 40x}{40y - 8y\left(x^2+y^2\right)}$$

at $(3,1)$ this is equal to -30. And hence the tangent line is $y = -30x + 91$.

3. Find the following limits:

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}=\lim_{x\to 0}\frac{2\sin 2x}{2x}=\lim_{x\to 0}\frac{2\cos 2x}{1}=2$$
$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0$$
$$\lim_{x\to 0^+}\frac{x}{\ln x}=0$$
$$\lim_{x\to\infty}\frac{\ln(\ln x)}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{\ln x}\frac{1}{x}}{\frac{1}{2}x^{-1/2}}=\lim_{x\to\infty}\frac{2x^{1/2}}{x\ln x}=\lim_{x\to\infty}\frac{2}{x^{1/2}\ln x}=0$$
$$\lim_{x\to\infty}x^2e^{-x}=\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0$$
$$\lim_{x\to 0}\frac{x\sin x}{x^2+1-\cos x}=\lim_{x\to 0}\frac{\sin x+x\cos x}{2x+\sin x}=\lim_{x\to 0}\frac{2\cos x-x\sin x}{2+\cos x}=\frac{2}{3}$$
$$\lim_{x\to 0}\frac{\cos x}{x}=\text{does not exist}$$
$$\lim_{x\to 0}\frac{\sin(ax)}{x}=\lim_{x\to 0}\frac{a\cos(ax)}{1}=a$$
$$\lim_{x\to 0}\frac{\sinh x}{x}=\lim_{x\to 0}\frac{\cosh x}{1}=1$$
$$\lim_{t\to 0}\frac{\cosh t^2-1}{t^4}=\lim_{t\to 0}\frac{2t\sinh t^2}{4t^3}=\lim_{t\to 0}\frac{\sinh t^2}{2t^2}==\lim_{t\to 0}\frac{2t\cosh t^2}{4t}=\frac{1}{2}$$

4. Give the following lines in slope/intercept form:

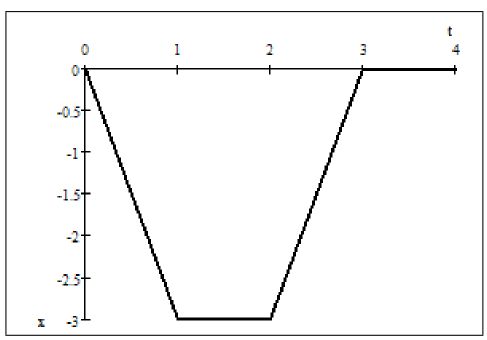
$x=3+4t,\ y=2+8t$ is $y=2x-4$

$x=-2+\frac{3}{4}t,\ y=2t$ is $y=\frac{8}{3}x+\frac{16}{3}$.

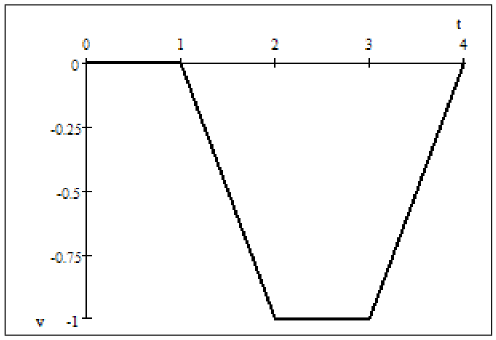
5. Give a parametrization of the circle of radius 4 centered at the origin which starts at (-4,0) and moves clockwise. There are many answers, such as $x=4\cos(\pi-t)$, $y=4\sin(\pi-t)$ with $t\in[0,2\pi]$ or $x=-4\cos t$, $y=-4\sin t$, $t\in[0,2\pi]$ or $x=4\cos(t)$, $y=-4\sin(t)$ $t\in[\pi,3\pi]$.

Sketch graphs of x vs t and y vs t for a parametrization of a rectangle whose vertices are (0,0), (-3, 0), (-3, -1), (0, -1) which starts at (0,0) and moves counterclockwise.

Plot of x vs t



Plot of y vs t:



6. Chapter 4 Check your understanding: 2) True 6) True 8) True 10) False, try x and $2x$. 14) $x(x-a)(x-b)$ for $a>0$ and $b>0$ 18)a) true (but not a best upper bound) b) false (the interval is open and it would be on the endpoint) c) false (same as last) d) false (it has one on, for instance (-2,2) e) true (true for any continuous function on a closed interval.

7. Chapter 4 Review: 2) Critical points at 2 (local and global min),3 (neither), global max at 5.
6) Increasing $(-\infty,-1)$ and $(1,\infty)$ and decreasing $(-1,1)$. Local max at -1 and local min at 1.
20) $\frac{dy}{dx} = a(1-bx)e^{-bx}$ so there is a critical point at $(2,10)$ if $a(1-2b)e^{-2b} = 0$, so $b = 1/2$ and $a2e^{-1} = 10$ so $a = 5e$. Note that at $(2,10)$, $\frac{d^2y}{dx^2} = -5/2$ so it is, in face, a local max.

8. a. Find two numbers whose sum is 100 and whose product is a maximum
Ans: both numbers are 50.
b. Find the points on the hyperbola $y^2 - x^2 = 4$ that are closest to the point $(2,0)$.
Ans: Minimize the square of the distance $f(x) = (x-2)^2 + \left(4+x^2\right)^2 = x^4+9x^2-4x+20$. $f'(x) = 4x^3+18x-4$. Critical point at $x = 0.219\,86$. $f''(x) = 12x^2+18$. Thus $f''(.22) = 12(.22)^2 + 18 = 18.\,581 > 0$ so the critical point is a minimum. The answers are $(2, \pm 2.012)$.
c. If 1200 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume.
Ans: Surface area $S = s^2 + 4sh = 1200$. The volume is $V = s^2h = \frac{s}{4}\left(1200 - s^2\right) = 300s - \frac{1}{4}s^3$. Thus

$$V'(x) = 300 - \frac{4}{3}s^2$$

and so the critical points are $s = \pm 15/2$, and the negative is outside the domain. Note that

$$V''(15/2) = -\frac{8}{3}\frac{15}{2} = -20 < 0$$

so this is a maximum. Largest volume is thus $V(15/2) = 300(15/2) - (1/4)(15/2)^3 = 2144.\,5$ cm^3.
d. A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into 4 pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the 4 pens.

Ans: The length of fencing will be $2w + 5l = 750$. The area is $A = lw$. Thus we minimize

$$A = \frac{1}{2}(750 - 5l)\,l$$

This is minimized at $l = 75$. And So the largest possible area is abouyt 14063 ft^2.
9. Find the best possible bounds for the following:
$f(x) = e^{-x^2}$ on $[-1,1]$

$f' = -2xe^{-x^2}$, so critical point at $x = 0$. So $f(0) = 1$, $f(1) = f(-1) - 1/e$. Thus

$$1/e \le e^{-x^2} \le 1$$

$\sin 4x$ on $[-\pi/2, \pi/6]$

Notice that there is an entire period on $[-\pi/2, \pi/6]$ and so the best bounds must be

$$-1 \le \sin 4x \le 1$$

$g(x) = Ae^{-x} + Be^{x}$ for $A > 0$ and $B > 0$.

$g'(x) = -Ae^{-x} + Be^{x}$. Critical points are at $e^{2x} = \frac{A}{B}$ which is positive, so there is a critical point at $x = \frac{1}{2}\ln\left(\frac{A}{B}\right)$. Note that $g''(x) = Ae^{-x} + Be^{x}$, so $g''\left(\frac{1}{2}\ln\left(\frac{A}{B}\right)\right) = A\sqrt{\frac{B}{A}} + B\sqrt{\frac{A}{B}} = 2\sqrt{\frac{A}{B}} > 0$ and so this is a minimum. Notice that

$$\lim_{x \to \infty} g(x) = \infty$$
$$\lim_{x \to -\infty} g(x) = \infty$$

and so there is no upper bound and the best possible lower bound is $g\left(\frac{1}{2}\ln\left(\frac{A}{B}\right)\right) = 2\sqrt{\frac{A}{B}}$.