Corrections/Clarifications on Logistics Model.

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Corrections about Logistics model.

1) We looked at the concavity of the graph of $\frac{dP}{dt}$ vs. P, which is a parabola facing down, and tried to draw conclusions about the graph of P vs. t. Our argument about $\frac{dP}{dt}$ as fine, but then we wanted to talk about concavity of $P(t)$, which basically requires us to look at the second derivative of P with respect to t. Notice that in regards to the graph of the parabola, where this is increasing/decreasing only tells us the sign of

$$\frac{d}{dP}\left(\frac{dP}{dt}\right).$$

In order to understand the concavity of P with respect to t, we need to use the chain rule, namely,

$$\frac{d^2P}{dt^2} = \frac{dP}{dt}\left[\frac{d}{dP}\left(\frac{dP}{dt}\right)\right].$$

Thus, when $P < 0$, then $\frac{dP}{dt} < 0$ and $\frac{d}{dP}\frac{dP}{dt} > 0$, so P is decreasing and concave down. When $0 < P < L/2$, we have $\frac{dP}{dt} > 0$ and $\frac{d}{dP}\frac{dP}{dt} > 0$, so P is increasing and concave up. When $L/2 < P < L$, we have $\frac{dP}{dt} > 0$ and $\frac{d}{dP}\frac{dP}{dt} < 0$, so P is increasing and concave down. Finally, when $P > L$, we have $\frac{dP}{dt} < 0$ and $\frac{d}{dP}\frac{dP}{dt} < 0$, so P is decreasing and concave up.

2) Let's re-derive the solution of the logistics equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right).$$

We separate variables,

$$\int \frac{1}{P\left(1-\frac{P}{L}\right)} dP = \int k dt$$

$$\int \left(\frac{1}{P} + \frac{1/L}{1-\frac{P}{L}}\right) dP = \int k dt$$

$$\ln\left|\frac{P}{1-P/L}\right| = kt + C$$

$$\frac{P}{1-P/L} = De^{kt}$$

$$P = \left(1-\frac{P}{L}\right) De^{kt}$$

$$P\left(1+\frac{1}{L}De^{kt}\right) = De^{kt}$$

$$P = \frac{De^{kt}}{1+\frac{1}{L}De^{kt}} = \frac{D}{e^{-kt}+\frac{D}{L}}.$$

To understand this solution, we need to rewrite D in terms of $P_0 = P(0)$. We see that

$$\frac{P_0}{1-P_0/L} = D,$$

so D is positive if $0 < P_0 < L$ and negative if $P_0 < 0$ or $P_0 > L$. Now, recall the solution is

$$P = \frac{De^{kt}}{1+\frac{1}{L}De^{kt}} = \frac{\frac{P_0}{1-P_0/L}}{e^{-kt}+\frac{P_0}{L-P_0}} = \frac{LP_0}{(L-P_0)e^{-kt}+P_0}$$

$$= \frac{LP_0}{Le^{-kt}+P_0\left(1-e^{-kt}\right)}.$$

Now, if $P_0 < 0$, we see that the numerator is always negative, and for very small values of t, the denominator is positive (since the second term is almost zero). However, as t increases, the first term gets very small, and the second term becomes more negative, so eventually the denominator becomes zero (for a finite value of t). Thus $P(t)$ goes to minus infinity along a vertical asymptote. We could actually find the t value for the asymptote, for it solves

$$Le^{-kt} + P_0\left(1-e^{-kt}\right) = 0$$

$$(L-P_0)e^{-kt} + P_0 = 0$$

$$e^{-kt} = \frac{P_0}{P_0 - L}$$

$$t = \frac{1}{k}\ln\left(\frac{P_0-L}{P_0}\right)$$

$$t = \frac{1}{k}\ln\left(1-\frac{L}{P_0}\right).$$

Since $P_0 < 0$, we see that $1 - L/P_0 > 1$ and this asymptote occurs when t is larger than zero.