Math 129-8H Project 2

Due Wednesday, November 26, in class

1 Part A:

We have seen in class that Taylor series can be derived from other Taylor series by substituting, integrating, and differentiating. In this part of the project, we will see how series can also be multiplied and divided.

1) We are quite good at multiplying polynomials. Multiplying Taylor series is much the same, but one must collect all the terms corresponding to the same power of x. Consider the following example:

$$\left(1+x+x^2+x^3+x^4+\cdots\right)\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots\right)=1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+\frac{65}{24}x^4+\cdots$$

where the terms come from multiplying the beginnings of the series and collecting terms. So we got the coefficients by:

$$\begin{aligned}
1 &= (1)(1) \\
2x &= (1)(x)+(x)(1) \\
\frac{5}{2}x^2 &= (1)\left(\frac{x^2}{2}\right)+(x)(x)+\left(x^2\right)(1) \\
\frac{8}{3}x^3 &= (1)\left(\frac{x^3}{6}\right)+(x)\left(\frac{x^2}{2}\right)+\left(x^2\right)(x)+\left(x^3\right)(1) \\
\frac{65}{24}x^4 &= (1)\left(\frac{x^4}{24}\right)+(x)\left(\frac{x^3}{6}\right)+\left(x^2\right)\left(\frac{x^2}{2}\right)+\left(x^3\right)(x)+\left(x^4\right)(1).
\end{aligned}$$

Note that we have computed the first few terms. It is quite difficult to find a general expression for the terms of a product of two series. Verify that the first two terms correspond to the first two terms of the Taylor series of the function

$$\frac{e^x}{1-x}$$

centered at $x=0$.

2) We have seen easy substitutions, but one can have more difficult substitutions. Suppose we wanted the Taylor series for $e^{\sin x}$. We could write the first few terms by seeing that

$$\begin{aligned}
e^{\sin x} &= 1+(\sin x)+\frac{1}{2}(\sin x)^2+\frac{1}{6}(\sin x)^3+\cdots \\
&= 1+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots\right)+\frac{1}{2}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^2+\frac{1}{6}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^3+\cdots \\
&= 1+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots\right)+\frac{1}{2}\left(x^2-2\frac{x^4}{3!}+\left(\frac{2}{5!}+\left(\frac{1}{3!}\right)^2\right)x^6+\cdots\right)+\frac{1}{6}\left(x^3-3\frac{x^5}{3!}+3\frac{x^7}{5!}\cdots\right)+\cdots \\
&= 1+x+\frac{x^2}{2}-\frac{x^3}{4}+\cdots
\end{aligned}$$

since all the later terms will have only x^4 terms or higher. We could continue to calculate terms (note that I calculated more terms than I had to in the square and the cube, just to demonstrate the process. Calculate the first 3 nonzero terms in the Taylor series of $\sin(\sin x)$. Explain why this will not work well for $\sin(\cos x)$. (Hint: what happens to the constant terms when they are squared? Do you get a finite or infinite sum?)

3) Using the series for $\frac{1}{1-x}$, one can also do division. Compute the first 3 nonzero terms of the series for

$$\frac{1}{\cos x}$$

by writing $\cos x$ as $1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$ and substituting $\left(\frac{x^2}{2} - \frac{x^4}{4!} + \cdots\right)$ into the series for $\frac{1}{1-x}$.

4) Calculate the first three nonzero terms in the Taylor series for $\tan x$ by multiplying the series in question 3 by the series for $\sin x$.

2 Part B:

In this part we will see that a series which does not converge absolutely cannot be added up in a different order. Recall that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges by the alternating series test. A rearrangement of a series is a reordering of all the terms so that we are still adding up the same terms, but they are in a different order. Why might it be possible that adding terms up in a different order may give a different result? (Hint: what does it mean for a series to converge?)

1) Show that the alternating harmonic series converges to a number less than $5/6$ (which is the partial sum S_3). (Hint: Show that the partial sums S_{2n+1} are decreasing. Since the sum must equal $\lim_{n\to\infty} S_{2n+1}$, we can find an estimate for the sum.)

2) Consider the following rearrangement of the alternating harmonic series:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots + \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} + \cdots$$

Notice that it contains all of the same terms as the alternating harmonic series, but just in a different order. Show that $\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0$ for any $k \geq 1$. Show that the sum is bigger than $5/6$. (Hint: the partial sums S_{3n} are increasing. Why?)

3) Note that you have just shown that the alternating harmonic series does not converge to the same value as a particular rearrangement! A surprising fact is that for any number R, we can find a rearrangement that converges to that number. This is a fact that one might prove in an upper division Real Analysis class.