CHAPTER 7: TECHNIQUES OF INTEGRATION

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1. Introduction

This semester we will be looking deep into the recesses of calculus. Some of the main topics will be:

- Integration: we will learn how to integrate functions explicitly, numerically, and with tables. You are expected already to have a concept of what an integral is (area under a function, sum of really small things, antiderivative). This includes both proper and improper integrals!
- Applications of integration: This topic so big and so fundamental, that we will barely scratch the surface. Only a small selection of topics will be covered:
  - Areas and volumes from slicing
  - Volumes of revolution
  - Length of curves
  - Density
  - Work and energy: consider the problem of the harmonic oscillator (spring)
  $$m\frac{d^2x}{dt^2} = -kx.$$
  What is the motion as a function of time t? of distance from equilibrium x?
- Sequences and series: How to you sum an infinite collection of numbers? It's not easy. How do you even know they have a sum? Can you use this to write down functions? Is this related to weird functions we already know (like sin, cos, log)?
- Complex numbers: You need to know the basics: addition, multiplication, division, polar form, roots
- Basic differential equations: This is a topic for another course or four, but we will learn the basics of what a differential equation is (hint: you already know this, but maybe haven't thought about it this way), the structure of solutions (how much information do you need to solve one?), how to solve the very easiest ones, a naive way to find numerical solutions, some super-duper basic applications like population growth.

This is the second course in calculus given here at UA. Some things to keep in mind:

- This course moves superfast. If you previously took calculus in high school or took Math 124 here, you will find that this class moves at least twice as fast. DO NOT BE LEFT BEHIND!

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- I will not be assigning enough homework for most of you to learn the material. Generally, all the homework in the assigned sections is relevant, so try to look at it all and see if you can do it. Learning mathematics is like riding a bike: you need to keep working at it until you "get it." Some people might just get right on the bike and start riding, others will fall a few times first. But you know when you've got it (or, at least, someone else does, and can tell you, "No, you are not riding that bike! You are just walking along the side and holding it up!") Maybe you need to get on from a different place or look at it a different way. Keep trying until you more or less get it. Sure, maybe you won't be ready to ride without your hands for a few more classes, but at least you can ride around the block to your friend's house and calculate some integrals.
- Did I mention this course moves superfast?
- Ask questions if you can (if you can't, it is likely you do not understand). Ask me in class. Ask me in office hours. Ask your friends. Ask tutors. What should you ask about? Ask "why." Ask about the meanings of symbols (they are strange that way for a reason, usually). Rephrase something in your own words and ask if that is the right idea.

2. Integration by substitution

We consider the problem of paramtrization. Compute an approximation of the integral

$$\int_0^4 \sqrt{2x+1}dx$$

by doing left and right sums with two subintervals:

$$LHS = (2)(1) + (2)\sqrt{5} = 2 + 2\sqrt{5}$$
$$RHS = 2\sqrt{5} + 2(3) = 6 + 2\sqrt{5}.$$

What if we change the variable to $y = 2x + 1$. Then the limits should go from $y(0) = 1$ to $y(4) = 9$ and the integrand is $\sqrt{y}$. Again, we compute the left and right sums with two subintervals:

$$LHS = (4)(1) + (4)\sqrt{5} = 4 + 4\sqrt{5}$$
$$RHS = 4\sqrt{5} + 4(3) = 12 + 4\sqrt{5}.$$

What do you notice? The problem is that we did not change dx to dy appropriately, and the change of interval is too large. We need to divide by two in order to do this appropriately. In integrals, this is

$$\int_0^4 \sqrt{2x+1}dx = \frac{1}{2}\int_1^9 \sqrt{y}dy.$$

The $\frac{1}{2}$ follows from the fact that we multiplied x by 2 to get y, so our intervals are twice as large and so we need to divide by two. Another way to think of this is

$$dy = 2dx = \frac{dy}{dx}dx.$$

The difference is whether we want to parametrize the integral by x or by y. It doesn't really matter in the end, since the definite integral is a number, not a function of x or of y.

This idea is very useful for calculating antiderivatives, as we will see soon. Here is another example of change of variables.

Consider a spring. Newton's law is that mass times acceleration equals force. Hooke's law is that force on a spring is proportional to displacement from equilibrium. If x denotes the displacement from equilibrium, we get the following equation governing the spring:

$$m\frac{d^2x}{dt^2} = -kx$$

where k is a positive constant. Now, suppose we wanted to solve this equation. Normally, we would integrate the equation with respect to t, but the force on the right is an explicit function of x, not t. Here is a very powerful idea:

$$\int_a^b m\frac{d^2x}{dt^2}dx = -\int_a^b kxdx.$$

The integral on the right is easy:

$$-\int_a^b kxdx = \frac{k}{2}\left(a^2 - b^2\right).$$

The integral on the left is harder, since it is written as a derivative of t, not x. However, by reparametrizing, we can write it as an integral over t:

$$\begin{aligned}
\int_a^b m\frac{d^2x}{dt^2}dx &= \int_{x=a}^{x=b} m\frac{d^2x}{dt^2}\frac{dx}{dt}dt \\
&= \int_{x=a}^{x=b} m\frac{dv}{dt}vdt \\
&= \int_{x=a}^{x=b} \frac{1}{2}m\left(\frac{d}{dt}v^2\right)dt \\
&= \frac{1}{2}m\left(v^2\big|_{x=b} - v^2\big|_{x=a}\right).
\end{aligned}$$

Now suppose we start at $a = L$ where velocity is zero (so we just release it). Then one way to denote the motion is by

$$\frac{1}{2}mv^2 = \frac{k}{2}\left(L^2 - x^2\right).$$

Where is the velocity zero? at $x = L$ or $x = -L$. Where is the velocity maximal? at $x = 0$.

Here is some easier stuff. You should have seen this last semester, so we will just be whizzing through this. Ask questions if you need to, though, since substitution is VERY, VERY, VERY important.

Example 1. *Calculate the antiderivative:*

$$\int 2xe^{x^2}dx$$

Solution 1. *Wait, don't look at this as a function of x, maybe use $w = x^2$ (the variable w is made up, and can be called whatever you like since it does not really exist until you make it up). Our goal is to replace all x's by w's. Be sure to change the dx to a dw!*

$$w = x^2 \qquad dw = 2x dx$$

Note: I like the differential notation, where $dw = \frac{dw}{dx} dx$, but the book does not (look it up!) Either is fine, as long as you use it properly. Okay, continuing...

$$\begin{aligned} \int 2xe^{x^2} dx &= \int e^w dw \\ &= e^w + C. \end{aligned}$$

But WAIT!!!!! The variable w did not exist before, so if we tell other people this is the answer, they won't know what it means! (It's like telling the police officer that your speed is really only 4 megabobs/hr. She'll write you a ticket before you can say "Buenos tardes") So we need to go back to x, and give

$$\int 2xe^{x^2} dx = e^{x^2} + C.$$

Done and done.

Solution 2. *Wait! Is this right? Let's check:*

$$\frac{d}{dx}\left(e^{x^2} + C\right) = 2xe^{x^2}.$$

Bingo.

Solution 3. *Maybe you were clever, and could see right away that the antiderivative should be $e^{x^2} + C$. That's fine, just check your guess ala the last solution. By the end of the week, you should be able to do that with an integral this easy, or you probably don't get it yet! In the end, you should be able to do this both ways!*

Example 2. *Find the antiderivative: $\int 3t^{99}\sqrt{t^{100} + 9}dt$.*

Solution 4. *Why do we use substitution? To get rid of stuff inside other stuff, usually. What's inside? It's that $t^{100} + 9$, that we don't know what to do with (remember, $\sqrt{t^{100} + 9} \neq t^{50} + 3$. That's just unbelievably wrong (do you really think that $\sqrt{10} = 4$? Seriously?) So*

$$w = t^{100} + 9 \quad dw = 100t^{99} dt.$$

Uh, oh. No $100t^{99}dt$ in there. But there is $3t^{99}dt$, so we adjust:

$$\begin{aligned} \int 3t^{99}\sqrt{t^{100} + 9}dt &= \int \sqrt{t^{100} + 9} 3t^{99} dt \\ &= \int \frac{3}{100}\sqrt{t^{100} + 9} 100t^{99} dt \\ &= \frac{3}{100}\int \sqrt{w} dw \\ &= \frac{1}{50} w^{3/2} + C \\ &= \frac{1}{50}\left(t^{100} + 9\right)^{3/2} + C \end{aligned}$$

Go ahead and check it. It's right.

Example 3. *Compute:* $\int_0^{\pi} \sin\theta \sin(\cos\theta)\, d\theta$

Solution 5. *Note that this is a definite integral! Okay, still, we have an inside function, and so*

$$w = \cos\theta \qquad dw = -\sin\theta d\theta$$

and so

$$\int_0^{\pi} \sin\theta \sin(\cos\theta)\, d\theta = -\int_0^{\pi} \sin(w)\, dw.$$

WAIT!!!!! Those bounds are not for w (my made up, imaginary variable friend), they are for θ! So, really we should have written

$$\int_0^{\pi} \sin\theta \sin(\cos\theta)\, d\theta = -\int_{0=\theta}^{\pi=\theta} \sin(w)\, dw.$$

This time, our answer should be a number, so it is perfectly okay to say, well, if $\theta = \pi$, *then* $w = \cos\pi = -1$, *and if* $\theta = 0$, *then* $w = \cos 0 = 1$, *and so*

$$\begin{aligned}\int_0^{\pi} \sin\theta \sin(\cos\theta)\, d\theta &= -\int_{0=\theta}^{\pi=\theta} \sin(w)\, dw \\ &= -\int_1^{-1} \sin(w)\, dw \\ &= \cos(-1) - \cos(1) \\ &(= 0 \text{ since } \cos(-1) = \cos(1))\end{aligned}$$

This answer does not have this mysterious w (since really θ was a dummy variable, too, since the integral gives a number, not a function), so it is perfectly kosher. OR, we could stick with θ, and do

$$\begin{aligned}\int_0^{\pi} \sin\theta \sin(\cos\theta)\, d\theta &= -\int_{0=\theta}^{\pi=\theta} \sin(w)\, dw \\ &= -\int_{0=\theta}^{\pi=\theta} \sin(w)\, dw \\ &= \cos w|_{0-\theta}^{\pi=\theta} \\ &= \cos(\cos\theta)|_{0-\theta}^{\pi=\theta} \\ &= \cos(-1) - \cos(1) \\ &(= 0 \text{ since } \cos(-1) = \cos(1))\end{aligned}$$

Example 4. $\int \frac{1}{4+y^2} dy$

Example 5. *Find the antiderivative:* $\int \tan\theta d\theta$

Example 6. *Compute:* $\int_0^1 \frac{dx}{2-x}$

Example 7. *Find the antiderivative:* $\int \sqrt{1-\sqrt{x}}dx$

Things to keep in mind:

- If you can't get rid of the dx or dt or $d\theta$ in the original problem, then you CANNOT use substitution!!!!!!!! VERY IMPORTANT. Very often it fails to work. For instance,

$$\int x^2 e^{x^2} dx$$

cannot be solved using substitution! You must turn everything into w, dw before you integrate. NEVER try to integrate something like $\int f(w)\,dx$. Ever.

- Get good at this. It should be one of the first tools you turn to. Plus, it's fun!
- Think about why some integrals can be done with substitution and some cannot. Sometimes you might be surprised.
- Be keenly aware of whether your answer should be a function of a particular variable (an antiderivative, a definite integral with a variable in the limits) or a number (a definite integral with numbers in the limits), etc.
- Think about all possible substitutions. For a given integral, there are only a few. Which ones work, which ones do not? Why?
- Remember, integration by substitution is all about replacing some piece of your integral with a new variable "blah" and rewriting your integral as integral of a function of "blah" d"blah". Don't forget the d"blah"!
- Substitution is the inverse of the chain rule. So when you check your answer, you'd better be using the chain rule!

3. Integration by parts

Integration by parts is a technique to use this really cool observation: By the product rule, we have

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

and so integrating with respect to x, we get

$$\int \frac{d}{dx}(uv)\,dx = \int \frac{du}{dx}v dx + \int u\frac{dv}{dx}dx.$$

So, for definite integrals, we have

$$uv|_a^b = \int_a^b \frac{du}{dx}v dx + \int_a^b u\frac{dv}{dx}dx$$

or, more compactly

$$uv|_{a=x}^{b=x} = \int_{a=x}^{b=x} v du + \int_{a=x}^{b=x} u dv$$

You can use this to turn one of the integrals on the right into another. Check it out:

$$\frac{d}{dx}(x\sin x) = \sin x + x\cos x$$

and so

$$x\sin x = \int \sin x dx + \int x\cos x dx,$$

so if I want to calculate

$$\begin{aligned}\int x\cos x dx &= x\sin x - \int \sin x dx \\ &= x\sin x + \cos x + C.\end{aligned}$$

Double check it!

In general, if we want to convert one integral to another using integration by parts, we need to do this:

$$\int u dv = uv - \int v du$$

and so you need to take your integral $\int f(x)\, dx$ and break it up into two pieces: u and dv, then use that to find du and v (so you differentiate u and integrate v). It only works if you can do that.

Example 8. *Find the antiderivative:* $\int t \ln t \; dt$

Example 9. *Find the antiderivative:* $\int y^2 e^y dy$

Example 10. *Compute:* $\int_0^x \cos^2 t \; dt$

Solution 6. *We do the following integration by parts:*

$$\begin{aligned} du &= \cos t \; dt \qquad v = \cos t \\ u &= \sin t \qquad dv = -\sin t \; dt \end{aligned}$$

to get

$$\begin{aligned} \int_0^x \cos^2 t \; dt &= \sin t \cos t|_0^x + \int_0^x \sin^2 t \; dt \\ &= \sin t \cos t|_0^x + \int_0^x \left(1 - \cos^2 t\right) \; dt \end{aligned}$$

and so

$$\begin{aligned} 2\int_0^x \cos^2 t \; dt &= \sin t \cos t|_0^x + \int_0^x \; dt \\ &= \sin x \cos x + x \end{aligned}$$

and so

$$\int_0^x \cos^2 t \; dt = \frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

Note: this is a definite integral, so there is no "+C."

Example 11. *Find the antiderivative:* $\int e^x \sin 3x \; dx$

Example 12. *Find the antiderivative:* $\int \arcsin x \; dx$

The basic idea: be sure to choose u so that it gets better when you differentiate it (natural log, polynomial, inverse trig) and v so that it doesn't get too worse when you integrate (exponential, sine, cosine)

4. Integration of rational functions

Rational functions have the form $\frac{P(x)}{Q(x)}$ where P and Q are polynomials. There are several techniques that we can use. First, let's look at when $P(x)$ is linear and $Q(x)$ is quadratic.

4.1. **Factoring and completing the square.** You will notice that sometimes the quadratic in the denominator looks like

$$\frac{****}{(x-a)(x-b)}$$

(where $a \neq b$) or

$$\frac{****}{x^2+a^2}$$

(where $a \neq 0$) and that is it (actually, there are the additional cases of the top case where $a = b$ and the bottom where $a = 0$, but these can be done directly with a substitution). In fact, the factoring (for quadratics) can be done using completing the square as well.

Completing the square is done as follows:

Example 13. *Complete the square to rewrite* $x^2 + 4x + 8$.

Solution 7. *We need to form a square, which gives* $\left(x^2+4x+4\right)+8-4=(x+2)^2+4$. *Note that this is* $y^2 + 2^2$, *where* $y = x + 2$.

Example 14. *Complete the square to rewrite* $x^2 + 4x + 3$.

Solution 8. *To form a square, we write* $\left(x^2+4x+4\right)+3-4=(x+2)^2-1^2=(x+2+1)(x+2-1)=(x+3)(x+1)$.

In general, one can complete the square for any quadratic polynomial x^2+bx+c by

$$\begin{aligned} x^2+bx+c &= x^2+bx+\left(\frac{b}{2}\right)^2+c-\left(\frac{b}{2}\right)^2 \\ &= \left(x+\frac{b}{2}\right)^2+\left(c-\left(\frac{b}{2}\right)^2\right). \end{aligned}$$

If $\left(c-\left(\frac{b}{2}\right)^2\right)$ is negative, then you can factor. If positive, then it is a square.

Note that they don't have to look so nice.

Example 15. *Complete the square to rewrite* $x^2 + 4x + 2$.

Solution 9. *To form a square, we write* $\left(x^2+4x+4\right)+2-4=(x+2)^2-\sqrt{2}^2=\left(x+2+\sqrt{2}\right)\left(x+2-\sqrt{2}\right)$.

Now, to do antiderivatives, we need to be able to integrate things of the form

$$\frac{****}{(x-a)(x-b)}$$

(where $a \neq b$) or

$$\frac{****}{x^2+a^2}$$

The first uses partial fractions and the second uses trig substition.

4.2. **Partial fractions.** Partial fractions is a way of undo-ing getting a common denominator. You are probably familiar with getting a common denominator:

$$\frac{1}{x+2} - \frac{1}{x-8} = \frac{-10}{(x+2)(x-8)}.$$

Conversely, we can start with the right side and try to get the left. How do we do this? One way is to just assume that this can be done, and solve for the right numbers:

$$\frac{-10}{(x+2)(x-8)} = \frac{A}{x+2} + \frac{B}{x-8}$$

for some choice of (numbers) A and B. Then we get a common denominator, or cross multiply, to get

$$-10 = A(x-8) + B(x+2).$$

This must be true FOR ALL VALUES OF x, and so we can plug in values of x until we can find A and B. If we are smart, we can save ourselves some work: try $x = 8$ and $x = -2$. Then we get

$$10B = -10$$

so $B = -1$ and

$$-10A = -10$$

and so $A = 1$. This tells us that

$$\frac{1}{x+2} - \frac{1}{x-8} = \frac{-10}{(x+2)(x-8)}.$$

We can always double check by getting a common denominator and seeing if this is right.

Remark 1. *The book does this slightly different, looking at the equation for A and B and equating the coefficients of x and the constant terms. This is, of course, correct, but my way is usually faster and just as correct.*

Now one can calculate the antiderivative

$$\begin{aligned}\int \frac{-10}{(x+2)(x-8)}dx &= \int \left(\frac{1}{x+2} - \frac{1}{x-8}\right) dx \\ &= \ln|x+2| - \ln|x-8| + C \\ &= \ln\left|\frac{x+2}{x-8}\right| + C.\end{aligned}$$

What more can we do with these kinds of problems:

- Maybe you need to factor the denominator. Consider the problem of finding the antiderivative
$$\int \frac{1}{x^2-2x-3}dx.$$
First we factor the denominator:
$$\frac{1}{x^2-2x-3} = \frac{1}{(x-3)(x+1)}.$$
Then we do partial fractions:
$$\begin{aligned}\frac{1}{x^2-2x-3} &= \frac{A}{x-3} + \frac{B}{x+1} \\ 1 &= A(x+1) + B(x-3)\end{aligned}$$

and so, using $x=-1$ and $x=3$, we get $B=-\frac{1}{4}$ and $A=\frac{1}{4}$, so

$$\frac{1}{x^2-2x-3}=\frac{1/4}{x-3}-\frac{1/4}{x+1}$$

and the antiderivative is

$$\begin{aligned}\int\frac{1}{x^2-2x-3}dx &= \frac{1}{4}\ln|x-3|-\frac{1}{4}\ln|x+1|+C\\ &= \ln\left|\frac{x-3}{x+1}\right|^{1/4}+C.\end{aligned}$$

- You can have polynomials in the numerator:

$$\frac{3x-1}{(x+2)(x+10)}.$$

  This works the same way, as long as the numerator has a lower degree than the denominator. If not, you need to do long division first!
- See book for the entire strategy for integrating a polynomial over another polynomial on p. 355.

4.3. **Trig substitution.** We can do an integral of a function of the form $\frac{x}{x^2+a^2}$ easily using a substition:

$$\int\frac{x}{x^2+a^2}dx=\frac{1}{2}\ln\left|x^2+a^2\right|+C$$

but for a function of the form $\frac{1}{x^2+a^2}$, it is harder. We have a very good trick. Try $x=a\tan\theta$. Then $dx=a\sec^2 x\ d\theta$ and we get

$$\begin{aligned}\int\frac{1}{x^2+a^2}dx &= \int\frac{a\sec^2 x}{a^2\tan^2\theta+a^2}d\theta\\ &= \int\frac{1}{a}d\theta\end{aligned}$$

since $\tan^2\theta+1=\sec^2\theta$. Hence

$$\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\arctan\frac{x}{a}+C.$$

Example 16. *Find the antiderivative:* $\int\frac{4x-5}{x^2+4x+3}dx$.

Solution 10. *Completing the square/factoring followed by partial fractions gives us*

$$\begin{aligned}\int\frac{4x-5}{x^2+4x+3}dx &= \int\frac{4x-5}{(x+3)(x+1)}dx\\ &= \int\left(\frac{17}{2(x+3)}-\frac{9}{2(x+1)}\right)dx\\ &= \frac{17}{2}\ln|x+3|-\frac{9}{2}\ln|x+1|+C\end{aligned}$$

:

Example 17. *Find the antiderivative:* $\int\frac{4x-5}{x^2+4x+8}dx$

Solution 11. *Completing the square, we get*

$$\int \frac{4x-5}{x^2+4x+8}dx = \int \frac{4x-5}{(x+2)^2+2^2}dx.$$

Now we can subsitution $w = x+2$, *to get the integral*

$$\int \frac{4x-5}{(x+2)^2+2^2}dx = \int \frac{4w-13}{w^2+2^2}dw$$

We now do the two parts separately. The first is a substitution $u = w^2+4$, $du = 2w\ dw$,

$$\int \frac{4w}{w^2+2^2}dw = \int \frac{2}{u}dw = 2\ln|u| + C = 2\ln\left|(x+2)^2+4\right| + C.$$

For the second, we can do a trig substitution $w = 2\tan\theta$, $dx = 2\sec^2\theta\ d\theta$ *to get*

$$\begin{aligned}\int \frac{13}{(x+2)^2+2^2}dx &= 13\int \frac{2\sec^2\theta}{4\tan^2\theta+4}d\theta \\ &= \frac{13}{2}\arctan\frac{x+2}{2} + C\end{aligned}$$

We arrive at

$$\int \frac{4x-5}{(x+2)^2+2^2}dx = 2\ln\left|(x+2)^2+4\right| - \frac{13}{2}\arctan\frac{x+2}{2} + C.$$

4.4. **Long division.** One may also need to do long division. If you have a quotient of polynomials $p(x)/q(x)$ and the degree on the top is not smaller than the degree on the bottom, you can divide. We will only use easy division of the type below:

Example 18. *Use division to simplify* $\frac{x^2}{x^2-1}$.

Solution 12. *The easiest is to just add and subtract to the top, instead of doing actual long division:*

$$\frac{x^2}{x^2-1} = \frac{x^2-1+1}{x^2-1} = \frac{x^2-1}{x^2-1} + \frac{1}{x^2-1} = 1 + \frac{1}{x^2-1}.$$

Example 19. *Use division to simplify* $\frac{x^3}{x^2-1}$.

Solution 13. *Same idea:*

$$\frac{x^3}{x^2-1} = \frac{x^3-x+x}{x^2-1} = \frac{x^3-x}{x^2-1} + \frac{x}{x^2-1} = x + \frac{x}{x^2-1}.$$

We can now compute antiderivatives using the table.

Example 20. *Find the antiderivative:* $\int \frac{x^2}{x^2-1}dx$

Solution 14. *Use partial fractions:*

$$\begin{aligned}\int \frac{x^2}{x^2-1}dx &= \int \left(1 + \frac{1}{x^2-1}\right)dx \\ &= \int \left(1 + \frac{1}{(x+1)(x-1)}\right)dx \\ &= \int \left(1 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right)dx \\ &= x + \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C\end{aligned}$$

Example 21. *Find the antiderivative:* $\int \frac{x^3}{x^2-1}dx$.

Solution 15. *Use partial fractions:*

$$\begin{aligned}\int \frac{x^3}{x^2-1}dx &= \int \left(x+\frac{x}{x^2-1}\right)dx \\ &= \int \left(x+\frac{x}{(x+1)(x-1)}\right)dx \\ &= \int \left(x+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}\right)dx \\ &= \frac{1}{2}x^2+\frac{1}{2}\left(\ln|x-1|+\ln|x+1|\right)+C\end{aligned}$$

4.5. **Partial fractions in more generality.**

- You can have more terms in the denominator, for instance
$$\frac{1}{(x+1)(x-9)(x+6)}.$$
In this case, you do the same thing:
$$\frac{1}{(x+1)(x-9)(x+6)}=\frac{A}{x+1}+\frac{B}{x-9}+\frac{C}{x+6}$$
and proceed.
- You may not have linear terms in the denominator:
$$\frac{1}{(x^2+1)(x-9)}$$
can be rewritten as
$$\frac{Ax+B}{x^2+1}+\frac{C}{x-9}.$$
and
$$\frac{1}{(x^4+8)(x+3)}$$
can be rewritten as
$$\frac{Ax^3+Bx^2+Cx+D}{x^4+8}+\frac{E}{x+3}.$$
- You can have powers in the denominator:
$$\frac{x^2-2}{(x+1)^2(x+6)}.$$
In this case you have to be careful:
$$\frac{x^2-2}{(x+1)^2(x+6)}=\frac{Ax+B}{(x+1)^2}+\frac{C}{(x+6)}$$
or you can consider this like
$$\frac{A(x+1)+D}{(x+1)^2}+\frac{C}{(x+6)}=\frac{A}{(x+1)}+\frac{D}{(x+1)^2}+\frac{C}{(x+6)}.$$
Let's do this one, since it is a bit more tricky:
$$\frac{x^2-2}{(x+1)^2(x+6)}=\frac{x^2-2}{(x+1)^2(x+6)}$$

so

$$x^2 - 2 = A(x+1)(x+6) + D(x+6) + C(x+1)^2.$$

So we can take $x = -1$ to get $-1 = 5D$, so $D = -1/5$. Now we take $x = -6$ to get $34 = 25C$, so $C = 34/25$. How do we find A? We just pick any value of x we want and use our C and D that we already found. I am partial to $x = 0$, so

$$\begin{aligned} -2 &= 6A - \frac{6}{5} + \frac{34}{25} \\ A &= -\frac{9}{25}. \end{aligned}$$

We get

$$\frac{x^2 - 2}{(x+1)^2 (x+6)} = \frac{-\frac{9}{25}}{(x+1)} - \frac{\frac{1}{5}}{(x+1)^2} + \frac{\frac{34}{25}}{(x+6)}.$$

We can also compute the antiderivative:

$$\begin{aligned} \int \frac{x^2 - 2}{(x+1)^2 (x+6)} dx &= \int \left(\frac{-\frac{9}{25}}{(x+1)} - \frac{\frac{1}{5}}{(x+1)^2} + \frac{\frac{34}{25}}{(x+6)} \right) dx \\ &= -\frac{9}{25} \ln|x+1| + \frac{1}{5(x+1)} + \frac{34}{25} \ln|x+6| + C \end{aligned}$$

4.6. **Trig substitution for roots.** What if we have something bad like

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

(and we don't remember that it is actually $\arcsin x + C$)? We can get a lot of mileage out of the identity:

$$\sin^2 x + \cos^2 x = 1$$

and its related relations

$$\begin{aligned} \tan^2 x + 1 &= \sec^2 x = \frac{1}{\cos^2 x} \\ 1 + \cot^2 x &= \csc^2 x = \frac{1}{\sin^2 x}. \end{aligned}$$

You see, if $x = \sin w$, then $\sqrt{1-x^2} = \sqrt{1 - \sin^2 w} = \cos w$, which is better. So let's try that as a substitution:

$$\begin{aligned} x &= \sin w \\ dx &= \cos w \; dw \end{aligned}$$

and so

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{1-\sin^2 w}} \cos w \; dw \\ &= \int dw \\ &= w + C \\ &= \arcsin w + C. \end{aligned}$$

Magic! Sometimes it works, and sometimes it doesn't. We just need to be slick. Notice that this kind of substitution looks like it is backwards from the other, as we write x as a function of w!

Here's a similar one:

$$\int \frac{1}{\sqrt{9-x^2}}dx.$$

This time we want to substitute $x = 3\sin w$. Try it.

What is this kind of integration good for? See this:

- Stuff with $\sqrt{a^2-x^2}$ in it. Try $x = a\sin w$.
- Stuff with a^2+x^2 or $\sqrt{a^2+x^2}$. Try $x = a\tan w$.
- Sometimes you need to be a bit more slick:
$$\int \frac{1}{\sqrt{16-25x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{\frac{16}{25}-x^2}}dx$$
and then you are good.
- Sometimes you need to complete the square to do this
$$\int \frac{1}{x^2+x+2}dx$$
$$\int \frac{1}{\sqrt{9+x-x^2}}dx$$

More examples:

$$\begin{aligned}\int \frac{1}{t^2\sqrt{t^2+1}}dt &= \int \frac{1}{\cos^2\theta\tan^2\theta\sqrt{\sec^2\theta}}d\theta\\ &= \int \frac{\cos\theta}{\sin^2\theta}d\theta\\ &= -\frac{1}{\sin\theta}+C\\ &= -\frac{1}{\sin\arctan t}+C\\ &= -\frac{\sqrt{1+t^2}}{t}+C\\ &= -\sqrt{\frac{1}{t^2}+1}+C\end{aligned}$$

$$\int \frac{1}{\sqrt{x^2-x+2}}dx$$

5. Integration by tables

You would think integrating using a table would be very easy, but you'd be wrong. The table only has a few forms, and so to use them, you need a few things:

- Be familiar with the table. The table often requires you to figure out the value of a parameter, for instance the table might have an entry like $\int \frac{1}{\cos^n x}dx$, so you need to figure out what n is in your particular problem. There may be restrictions on n, like it must be positive or even or something. Be careful.

- The table entries may require you to understand a fairly large sum. Try to understand the entries by doing examples.
- You may need to transform your integral to get it to look like a table entry. Usually this means substitution, factoring, completing the square, long division, sometimes others.

Example 22. *Find the antiderivative:* $\int \left(x^3 - 2x + 6\right) \sin 3x \ dx$

Example 23. *Find the antiderivative:* $\int \cos^5 x \ dx$

There are some more techniques that are pretty useful:

Example 24. *Find the antiderivative:* $\int \frac{1}{\sqrt{x^2+6x+8}} dx$

5.1. **Summary.** Things to keep in mind:

- Be careful what your variable name is and if you need to do a substitution!
- Be careful with reading the table, especially to see which formula applies.

6. Numerical integration

Quite often you will find that you need to calculate definite integrals even if you cannot compute explicit expression. We have already seen left and right Riemann sums. These are generally poor, as you can see graphically. To correct them, we have two approaches:

1) Midpoint rule: use the midpoint instead of the left or right endpoints.

2) Trapezoid rule: average the left and right sums.

Both of these give much better approximations to the integral!

Example: $\int_0^1 x^2 dx = \frac{1}{3}$. Now let's compute this with left, right, midpoint, trapezoid rules

http://www.zweigmedia.com/RealWorld/integral/integral.html
http://www.hostsrv.com/webmaa/app1/MSPScripts/webm1010/simpson.jsp