Chapter 9 and 10 Review

1) Consider the series $\sum_{n=0}^{\infty} a_n$. State if the following are true or false. Be sure to justify your answers.

a) If $\lim_{n\to\infty} \sum_{j=0}^{n} a_n = S$ then $\sum_{n=0}^{\infty} a_n = S$.

b) If $\lim_{n\to\infty} a_n = 0$ then the series converges.

c) If the series is a geometric series and $\lim_{n\to\infty} a_n = 0$ then the series converges.

d) If the series is alternating and $\lim_{n\to\infty} a_n = 0$ then the series converges.

e) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then the series converges.

f) If $\lim_{n\to\infty} a_n < 1$ then the series converges.

g) If $a_n < \frac{1}{n^2}$ then the series converges.

h) If $|a_n| \leq \frac{1}{n}$ then the series converges.

i) If $|a_n| \geq \frac{1}{n}$ then the series diverges.

j) If $\sum_{n=0}^{\infty} a_n$ diverges and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} (a_n + b_n)$ diverges.

k) If $\sum_{n=0}^{\infty} |a_n|$ diverges then $\sum_{n=0}^{\infty} a_n$ diverges.

l) If the power series $\sum_{n=0}^{\infty} b_n (x-4)^n$ has radius of convergence 5 then the series converges for $-5 \leq x \leq 5$.

2) Find the Taylor series for the following functions and give the radius of convergence and the interval of convergence.

a) $\sin(x^3)$ at $x = 0$. b) $\cos 3x$ at $x = 0$. c) $x(e^x - 1)$ at $x = 0$.

d) $\frac{1}{x}$ at $x = 1$. e) $\arctan\left(\frac{x}{4}\right)$ at $x = 0$. f) $\ln(-x)$ at $x = -1$.

g) $x^4 + 2x$ at $x = 0$. h) $x^4 + 2x$ at $x = 2$. f) $\frac{1}{(1-x)^3}$ at $x = 0$.

3) Compute the following:

a) $\sum_{n=0}^{20} 3\left(\frac{5}{2}\right)^n$ b) $\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^n$ c) $\sum_{n=2}^{\infty} \left(\frac{3}{2}\right)^n$

4) Find if the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ b) $\sum_{n=1}^{\infty} \frac{1}{5n}$ c) $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$ d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ e) $\sum_{n=10}^{\infty} \frac{10}{n^{10}}$

f) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ g) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ h) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$ i) $\sum_{n=2}^{\infty} e^{-n}$

j) $1 - 1 + 1 - 1 + 1 - 1 + 1 \cdots$ k) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \cdots$

l) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$ m) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n}}$

5) Find the following by recognizing the appropriate Taylor series.

a) $1 + (0.1)^2 + (0.1)^3 + (0.1)^4 + (0.1)^5 + \cdots$ b) $\sum_{n=1}^{\infty} \frac{(0.3)^n}{n}$

c) $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \cdots$ d) $1 + x^2 + x^4 + x^6 + x^8 + \cdots$

e) $\sum_{n=1}^{\infty} 2^n \frac{x^n}{n!}$ f) $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!}$

6. Estimate the errors of the following approximations.

a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ as an approximation to $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$.

b) $1 - \frac{(0.5)^2}{2} + \frac{(0.5)^4}{4!}$ as an approximation to $\cos(0.5)$.

7. What degree Taylor polynomial can we use to approximate the following to 5 decimal places?

a) $\sin(1)$ b) $\frac{1}{1-(1/\sqrt{2})}$ (note that $0.5 < 1/\sqrt{2} < 1$) c) $\pi = 4\arctan(1)$

8. Compute the radius of convergence for the following power series. Also give the interval of convergence if possible.

a) $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)^2}(x-3)^n$ b) $\sum_{n=0}^{\infty} \frac{9^n (n+1)^3}{n!}(x+1)^n$ c) $\sum_{n=3}^{\infty} n(-3)^n x^{2n+1}$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3}(x-4)^{2n}$

9. Use Taylor series to find the following limits:

a) $\lim_{x\to 0} \frac{\sin x}{\ln(1-x)}$ b) $\lim_{x\to 0} \frac{\sin(x^2)}{\sin^2 x}$