

Chapter 9 and 10 Review

1) Consider the series  $\sum_{n=0}^{\infty} a_n$ . State if the following are true or false. Be sure to justify your answers.

a) If  $\lim_{n \rightarrow \infty} \sum_{j=0}^n a_j = S$  then  $\sum_{n=0}^{\infty} a_n = S$ .

True: definition of convergence.

b) If  $\lim_{n \rightarrow \infty} a_n = 0$  then the series converges.

False: consider  $a_n = 1/n$ .

c) If the series is a geometric series and  $\lim_{n \rightarrow \infty} a_n = 0$  then the series converges.

True: this means that the ratio of the geometric series is less than 1 and hence the series converges.

d) If the series is alternating and  $\lim_{n \rightarrow \infty} a_n = 0$  then the series converges.

True: this is the alternating series test.

e) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$  then the series converges.

False: the ratio test is inconclusive. Take, for instance,  $a_n = 1/n$  (diverges) and  $a_n = 1/n^2$  (converges).

f) If  $\lim_{n \rightarrow \infty} a_n < 1$  then the series converges.

False: if the limit is not zero, it will not converge (and may not even if it does equal zero).

g) If  $a_n < \frac{1}{n^2}$  then the series converges.

False if  $a_n$  is large and negative. This is true if  $a_n \geq 0$  by the comparison test.

h) If  $|a_n| \leq \frac{1}{n}$  then the series converges.

False:  $a_n = 1/n$  or  $a_n = 1/(2n)$  do not converge.

i) If  $|a_n| \geq \frac{1}{n}$  then the series diverges.

False: if  $a_n = (-1)^n \frac{1}{n}$  then it converges.

j) If  $\sum_{n=0}^{\infty} a_n$  diverges and  $\sum_{n=0}^{\infty} b_n$  diverges, then  $\sum_{n=0}^{\infty} (a_n + b_n)$  diverges.

False: try  $a_n = 1/n$  and  $b_n = -1/n$ .

k) If  $\sum_{n=0}^{\infty} |a_n|$  diverges then  $\sum_{n=0}^{\infty} a_n$  diverges.

False: take  $a_n = (-1)^n / n$ .

l) If the power series  $\sum_{n=0}^{\infty} b_n (x - 4)^n$  has radius of convergence 5 then the series converges for  $-5 \leq x \leq 5$ .

False: it converges for  $|x - 4| < 5$  or  $-1 < x < 9$  (it may also converge at one or both of the endpoints  $x = -1$  and  $x = 9$ , but we don't have enough information to find out).

2) Find the Taylor series for the following functions and give the radius of convergence and the interval of convergence.

a)  $\sin(x^3)$  at  $x = 0$ .

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{3(2n+1)}}{(2n+1)!}$$

b)  $\cos 3x$  at  $x = 0$ .

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n} x^{2n}}{(2n)!}$$

c)  $x(e^x - 1)$  at  $x = 0$ .

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n!}$$

d)  $\frac{1}{x}$  at  $x = 1$ .

$$\sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

e)  $\arctan\left(\frac{x}{4}\right)$  at  $x = 0$ .

$$\int \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{4}\right)^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{2n} (2n+1)} x^{2n+1}$$

f)  $\ln(-x)$  at  $x = -1$ .

$$\sum_{n=1}^{\infty} \left(-\frac{1}{n}\right) (x+1)^n$$

g)  $x^4 + 2x$  at  $x = 0$ .

$$2x + x^4$$

h)  $x^4 + 2x$  at  $x = 2$ .

$$20 + 34(x-2)^1 + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4$$

:

f)  $\frac{1}{(1-x)^3}$  at  $x = 0$ .

$$\begin{aligned} \frac{1}{(1-x)^3} &= \frac{1}{2} \frac{d^2}{dx^2} \frac{1}{(1-x)} \\ &= \frac{1}{2} \frac{d^2}{dx^2} \sum_{n=0}^{\infty} x^n \\ &= \frac{1}{2} \frac{d}{dx} \sum_{n=1}^{\infty} nx^{n-1} \\ &= \frac{1}{2} \sum_{n=2}^{\infty} n(n-1)x^{n-2} \\ &= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n \end{aligned}$$

3) Compute the following:

a)  $\sum_{n=0}^{20} 3 \left(\frac{5}{2}\right)^n = \frac{3(1-(\frac{5}{2})^{21})}{1-\frac{5}{2}} = 2 \left( \left(\frac{5}{2}\right)^{21} - 1 \right)$

b)  $\sum_{n=1}^{\infty} 2 \left(\frac{1}{3}\right)^n = \frac{2/3}{1-1/3} = 1$

c)  $\sum_{n=2}^{\infty} \left(\frac{3}{2}\right)^n$  diverges.

4) Find if the following series converge or diverge.

a)  $\sum_{n=1}^{\infty} \frac{n^2}{n!}$  converges by the ratio test ( $\lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(n+1)!} \frac{n!}{n^2} \right| = 0 < 1$ ).

b)  $\sum_{n=1}^{\infty} \frac{1}{5n}$  diverges (it is  $\frac{1}{5}$  the harmonic series).

c)  $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$  converges by ratio test ( $\lim_{n \rightarrow \infty} \left| \frac{n+2}{2^{n+1}} \frac{2^n}{n+1} \right| = \frac{1}{2} < 1$ ).

d)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  diverges by p-series (or integral test)

e)  $\sum_{n=10}^{\infty} \frac{10}{n^{10}}$  converges by p-series (or integral test)

f)  $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$  converges (compare the absolute value to  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  and hence the absolute value series  $\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n} \right|$  converges. Hence the original series converges.

g)  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$  converges by ratio test ( $\lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = 0$ )

h)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$  converges by the alternating series test since  $\lim_{n \rightarrow \infty} \frac{1}{3n} = 0$ .

i)  $\sum_{n=2}^{\infty} e^{-n}$  converges because it is a geometric series with ratio  $\frac{1}{e} < 1$ .

j)  $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$  diverges because the partial sums do not converge.

k)  $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \dots$  converges because it is a geometric series with ratio  $= -\frac{2}{3}$ , which is less than 1 in absolute value.

l)  $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$  diverges (harmonic series or use integral test)

m)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n}}$  converges (compare to  $\sum_{n=1}^{\infty} n^{-3/2}$  which converges by p-series or integral test)

5) Find the following by recognizing the appropriate Taylor series.

a)  $1 + (0.1)^2 + (0.1)^3 + (0.1)^4 + (0.1)^5 + \dots = \frac{1}{1-0.1} - 0.1 = \frac{10}{9} - \frac{1}{10} = \frac{91}{90}$

b)  $\sum_{n=1}^{\infty} \frac{(0.3)^n}{n} = -\ln(1 - 0.3) = -\ln(0.7)$

c)  $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \dots = 1 - e^{-1}$

d)  $1 + x^2 + x^4 + x^6 + x^8 + \dots = \frac{1}{1-x^2}$

e)  $\sum_{n=1}^{\infty} 2^n \frac{x^n}{n!} = e^{2x} - 1$

f)  $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!} = x - \sin x$

6. Estimate the errors of the following approximations.

a)  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$  as an approximation to  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ .

The error is less than  $\frac{1}{6}$  (alternating series approximation).

b)  $1 - \frac{(0.5)^2}{2} + \frac{(0.5)^4}{4!}$  as an approximation to  $\cos(0.5)$ .

The error is less than  $(1) \frac{(0.5)^6}{6!} = 2.1701 \times 10^{-5}$  (since the sixth derivative of  $\cos x$  is bounded by 1).

7. What degree Taylor polynomial can we use to approximate the following to 5 decimal places?

a)  $\sin(1)$

The error of the  $n$ th degree polynomial approximation is less than  $\frac{1}{n!}$ , so for 5 decimal places we need the error to be less than  $0.5 \times 10^{-5}$ , so we need 9th degree.

b)  $\frac{1}{1-(1/\sqrt{2})}$  (note that  $0.5 < 1/\sqrt{2} < 0.9$ )

In this case it is difficult to use the Taylor series estimate on error, but we do know that the error of the  $n$ th degree Taylor polynomial  $T_n$  is

$$\frac{1}{1-(1/\sqrt{2})} - T_n = \sum_{i=n+1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^i = \left(\frac{1}{\sqrt{2}}\right)^{n+1} \frac{1}{1-1/\sqrt{2}}.$$

Hence if  $n = 38$  we get an error of less than  $0.5 \times 10^{-5}$ .

c) We cannot approximate  $\arctan(1)$  by a Taylor series centered at  $x = 0$  (it does not converge).

8. Compute the radius of convergence for the following power series. Also give the interval of convergence if possible.

a)  $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)^2} (x-3)^n$

Radius of convergence = 1, Interval of convergence =  $(2, 4)$ .

b)  $\sum_{n=0}^{\infty} \frac{9^n (n+1)^3}{n!} (x+1)^n$

Radius of convergence =  $\infty$ , Interval of convergence =  $(-\infty, \infty)$ .

c)  $\sum_{n=3}^{\infty} n (-3)^n x^{2n+1}$

Radius of convergence =  $1/\sqrt{3}$ , Interval of convergence =  $(-1/\sqrt{3}, 1/\sqrt{3})$ .

d)  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3} (x-4)^{2n}$

Radius of convergence = 1, Interval of convergence =  $[3, 5]$ .

9. Use Taylor series to find the following limits:

a)  $\lim_{x \rightarrow 0} \frac{\sin x}{\ln(1-x)} = -1$

b)  $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{\sin^2 x} = 1$