

Review for Test 1

1. From Chapter 7 Check Your Understanding (p. 342): 3, 10, 11, 12, 13, 14, 15, 20, 21

Only answers to even numbers are given. Answers to odd numbers are in the back of the book.

10) True. Do the substitution $u = f(x)$

12) True. Do the substitution $u = 5 - t^2$

14) False. It will involve an arctan since you must complete the square (if it factored, it would involve a logarithm)

20) False. Take, for instance, $\int_0^\infty \frac{1}{\sqrt{x+1}} dx$.

2. From Chapter 7 Review (pp. 338-341): 5, 11, 12, 15, 17, 22, 25, 38, 43, 58, 64, 77, 79, 83, 86, 87, 89, 91, 102, 112, 119, 121, 123, 127, 128, 130, 133, 147

Only answers to even numbers are given. Answers to odd numbers are in the back of the book.

12)

$$\int \frac{(1 + \ln x)^2}{x} dx = \frac{1}{3} (1 + \ln x)^3 + C$$

by substitution.

22)

$$\int \sin^2 \theta \cos \theta d\theta = \frac{1}{3} \sin^3 \theta + C$$

by substitution.

38)

$$\int \frac{(\ln x)^2}{x} dx = \frac{1}{3} (\ln x)^3 + C$$

by substitution

58)

$$\int \frac{\cos w}{1 + \sin^2 w} dw = \arctan(\sin w) + C$$

by substitution or trig substitution or both.

64)

$$\int \frac{\sin w}{\sqrt{1 - \cos w}} dw = 2\sqrt{1 - \cos w} + C$$

by substitution.

86)

$$\int \frac{x^2 + 1}{x^2 - 3x + 2} dx = x - 2 \ln |x - 1| + 5 \ln |x - 2| + C$$

by division and partial fractions (or division and the table entry).

102)

$$\int \frac{\sqrt{\sqrt{x+1}}}{\sqrt{x}} dx = \frac{4}{3} (\sqrt{x+1})^{3/2} + C$$

by substitution.

112)

$$\int \cos(2 \sin x) \cos x dx = \frac{1}{2} \sin(2 \sin x) + C$$

by substitution.

128)

$$\int_{10}^{\infty} \frac{dx}{x \ln x}$$

diverges.

130)

$$\int_{-1}^1 \frac{1}{x^4} dx$$

diverges.

3. Write

$$\frac{x^3 + 3x}{(x + 2)(x - 1)}$$

in the form

$$p(x) + \frac{A}{x + 2} + \frac{B}{x - 1}$$

where $p(x)$ is a polynomial and A and B are numbers.

First we must do long division:

$$\begin{array}{r}
 x^2 + x - 2 \quad \overline{) \begin{array}{r} x^3 + 0x^2 + 3x + 0 \\ - (x^3 + x^2 - 2x + 0) \\ \hline -x^2 + 5x + 0 \\ - (-x^2 - x + 2) \\ \hline 6x - 2 \end{array} \\
 \end{array}$$

so we have

$$\frac{x^3 + 3x}{(x + 2)(x - 1)} = x - 1 + \frac{6x - 2}{(x + 2)(x - 1)}$$

We now compute partial fractions as

$$A(x - 1) + B(x + 2) = 6x - 2$$

So taking $x = 1$ we get

$$\begin{aligned}
 3B &= 4 \\
 B &= \frac{4}{3}
 \end{aligned}$$

and taking $x = -2$ we get

$$\begin{aligned} -3A &= -14 \\ A &= \frac{14}{3} \end{aligned}$$

and the answer is

$$\frac{x^3 + 3x}{(x+2)(x-1)} = x - 1 + \frac{14/3}{x+2} + \frac{4/3}{x-1}$$

We can check our answer by getting a common denominator

$$\begin{aligned} x - 1 + \frac{14/3}{x+2} + \frac{4/3}{x-1} &= \frac{(x-1)(x+2)(x-1) + 14/3(x-1) + 4/3(x+2)}{(x+2)(x-1)} \\ &= \frac{x^3 + 3x}{(x+2)(x-1)} \end{aligned}$$

4. Show that $\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$ for any positive n [hint: write $\sin^n x$ as $(\sin^{n-1} x)(\sin x)$].

Since we want to reduce the degree of $\sin^n x$ and based on the hint which shows a product, we can try integration by parts. If we take $u = \sin^{n-1} x$, $dv = \sin x \, dx$ (so $du = (n-1) \sin^{n-2} x \cos x \, dx$, $v = -\cos x$) we get

$$\begin{aligned} \int \sin^n x \, dx &= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x \, dx \\ &= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x (1 - \sin^2 x) \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx \end{aligned}$$

Now we can solve for $\int \sin^n x \, dx$

$$n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx.$$

Dividing by n gives the answer.

5. Compute $\int \sin(\ln y) \, dy$.

There are two ways to do this problem. The first is to do integration by parts. Take $u = \sin(\ln y)$, $dv = dy$ (so $du = \cos(\ln y) \frac{1}{y} dy$ and $v = y$) to get

$$\int \sin(\ln y) \, dy = y \sin(\ln y) - \int \cos(\ln y) \, dy.$$

Note that if we want to do integration by parts, this is really the only possibility since we cannot integrate the whole thing and there is no real product (this is similar to when we used integration by parts to compute $\int \ln x \, dx$). We are

maybe encouraged by the fact that the integral we still have to do looks similar to the original one. Let's give the integrals names to make things easier; let

$$I_1 = \int \sin(\ln y) dy$$

$$I_2 = \int \cos(\ln y) dy$$

so the answer, I_1 is computed as

$$I_1 = y \sin(\ln y) - I_2.$$

Let's do integration by parts again to compute I_2 . Let $u = \cos(\ln y)$, $dv = dy$ (so $du = -\sin(\ln y) \frac{1}{y} dy$ and $v = y$) and get

$$\begin{aligned} I_2 &= y \cos(\ln y) + \int \sin(\ln y) dy \\ &= y \cos(\ln y) + I_1 + C \end{aligned}$$

We now see that

$$\begin{aligned} I_1 &= y \sin(\ln y) - I_2 \\ &= y \sin(\ln y) - (y \cos(\ln y) + I_1 + C) \end{aligned}$$

so solving for I_1 we get

$$I_1 = \frac{1}{2} (y \sin(\ln y) - y \cos(\ln y)) + C$$

We can check our answer very easily, as

$$\begin{aligned} \frac{d}{dy} \left[\frac{1}{2} (y \sin(\ln y) - y \cos(\ln y)) \right] &= \frac{1}{2} [\sin(\ln y)] + y \cos(\ln y) \frac{1}{y} \\ &\quad - \cos(\ln y) + y \sin(\ln y) \frac{1}{y} \\ &= \sin(\ln y). \end{aligned}$$

An alternative method is to make the substitution $w = \ln y$ (so $dw = \frac{1}{y} dy$) and we see that

$$\begin{aligned} \int \sin(\ln y) dy &= \int \sin(w) y \left(\frac{1}{y} dy \right) \\ &= \int \sin(w) y dw \\ &= \int e^w \sin w dw \end{aligned}$$

since $w = \ln y$ implies that $y = e^w$. This integral can be solved either using tables or integrating by parts twice as done in lecture.

6. Suppose $f(1) = 5, f(0) = -1, \int_0^1 f(t) dt = 3$. Then compute:

a. $\int_{-1}^0 f(x+1) dx$

Make the substitution $u = x + 1$ (so $du = dx$) and we get

$$\begin{aligned}\int_{-1}^0 f(x+1) dx &= \int_{-1=x}^{0=x} f(u) du \\ &= \int_0^1 f(u) du \\ &= 3\end{aligned}$$

since if $x = -1$ then $u = x + 1 = -1 + 1$ and if $x = 0$ then $u = 0 + 1 = 1$.

b. $\int_{1/2}^1 f(2x-1) dx$

Make the substitution $u = 2x - 1$ (so $du = 2dx$) and we get

$$\begin{aligned}\int_{1/2}^1 f(2x-1) dx &= \int_{1/2=x}^{1=x} f(u) \frac{1}{2} du \\ &= \frac{1}{2} \int_0^1 f(u) du \\ &= \frac{3}{2}.\end{aligned}$$

c. $\int_0^1 xf(x^2) dx$

Make the substitution $u = x^2$, so $du = 2xdx$ and we get

$$\begin{aligned}\int_0^1 xf(x^2) dx &= \int_0^1 f(u) \frac{1}{2} du \\ &= \frac{3}{2}\end{aligned}$$

since if $x = 0$ then $u = x^2 = 0^2 = 0$ and if $x = 1$ then $u = 1^2 = 1$.

d. $\int_0^1 xf'(x) dx$

Integrate by parts taking $u = x$, $dv = f'(x) dx$ (so $du = dx$ and $v = f(x)$) and we get

$$\begin{aligned}\int_0^1 xf'(x) dx &= (xf(x))\Big|_0^1 - \int_0^1 f(x) dx \\ &= 5 - 0 - 3 \\ &= 2.\end{aligned}$$

e. $\int_0^1 \frac{f'(x)}{1+[f(x)]^2} dx$

There are several ways to do this problem. One is to make the

substitution $w = f(x)$ (so $dw = f'(x) dx$) and get

$$\begin{aligned} \int_0^1 \frac{f'(x)}{1 + [f(x)]^2} dx &= \int_{0=f(0)}^{1=f(1)} \frac{1}{1 + w^2} dw \\ &= \int_{f(0)}^{f(1)} \frac{1}{1 + w^2} dw \\ &= \arctan(f(1)) - \arctan(f(0)) \\ &= \arctan 5 - \arctan(-1). \end{aligned}$$

The last integral can be done by trigonometric substitution $w = \tan \theta$ if not known. The other way is to start out trying trigonometric substitution of the form $f(x) = \tan \theta$ (so $f'(x) dx = \frac{1}{\cos^2 \theta} d\theta$).

7. Suppose $f(x)$ is an increasing, concave down function. Put LEFT(200), RIGHT(200), MID(200), TRAP(200), and SIMP(200) in order from smallest to largest.

$$\text{LEFT}(200) < \text{TRAP}(200) < \text{SIMP}(200) < \text{MID}(200) < \text{RIGHT}(200).$$

8. Compute the exact value of $\int_0^1 \frac{4}{1+t^2} dt$ as well as LEFT(100), RIGHT(100), MID(100), TRAP(100), and SIMP(100). How many decimal places of accuracy does each achieve?

The exact value is $4(\arctan(1) - \arctan(0)) = \pi$. Using Allsums, we get

$$\begin{aligned} \text{LEFT}(100) &= 3.151575987 \\ \text{RIGHT}(100) &= 3.131575987 \\ \text{MID}(100) &= 3.141575987 \\ \text{TRAP}(100) &= 3.141600987 \\ \text{SIMP}(100) &= 3.141592654 \end{aligned}$$

Since π is approximately 3.141592654, we see that LEFT(100) and RIGHT(100) are both accurate to 1 decimal place, MID(100) and TRAP(100) are accurate to 4 decimal places, and SIMP(100) is accurate to at least 8 decimal places.

9. Suppose $f(x)$ is an increasing function and you know that LEFT(100)=0.123 and MID(100)=0.234. What can you say about the value of the integral (for instance, can you find an upper and/or lower bound)? Can you give an upper bound on the absolute value of the error?

Since the function is increasing, we know the left sum is a lower bound. The midpoint rule estimate does not help us bound the function at all and so all we can say is that the value of the integral is larger than 0.123. We cannot bound the error.

10. Suppose $f(x)$ is a decreasing, concave up function and you know that LEFT(100)=0.234 and MID(100)=0.123. What can you say about the value of the integral (for instance, can you find an upper and/or lower bound)? Can you give an upper bound on the absolute value of the error?

Since the function is decreasing, the left sum is an upper bound. Since it is concave up, the midpoint sum is a lower bound. Hence the integral must be between 0.123 and 0.234. The error of either of these approximations must be less than $(0.234-0.123)=0.111$.

11. Compute $\int_{-1}^1 \sqrt{1-x^2} dx$ using trigonometric substitution and the fact that $\sin^2 \theta = \frac{1-\cos 2\theta}{2}$ or $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$. Geometrically, what should the answer be?

Geometrically, this is the area of the upper half of a circle of radius 1 centered at the origin, so it should be $\frac{1}{2}\pi (1)^2 = \frac{\pi}{2}$. We can use the substitution $x = \sin \theta$ (so $dx = \cos \theta d\theta$) and get

$$\begin{aligned} \int_{-1}^1 \sqrt{1-x^2} dx &= \int_{-1=x}^{1=x} \sqrt{1-\sin^2 \theta} \cos \theta d\theta \\ &= \int_{-1=x}^{1=x} \cos^2 \theta d\theta \\ &= \int_{-\pi/2=x}^{\pi/2=\theta} \frac{1+\cos 2\theta}{2} d\theta \\ &= \left(\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta \right) \Big|_{-\pi/2}^{\pi/2} \\ &= \frac{\pi}{2}. \end{aligned}$$