Review for Test 2

1. Chapter 7 Check Your Understanding: 18, 20, 25,27
   18. True
   20. False. Try $f(x) = \frac{1}{x+1}$.
   25. True. Substitute $w = ax$.
   27. False.

2. Chapter 8 Check Your Understanding: 1, 2, 3, 4,13, 15
   1. True
   2. False
   3. False
   4. True
   13. False
   15. False

3. Chapter 9 Check Your Understanding: 14, 22, 23, 24, 25
   14. False. The terms do not go to zero.
   22. True.
   23. False. Consider the harmonic series.
   24. False. Consider $a_n = b_n = \frac{1}{n}$.
   25. False. Same example as 24.

4. Chapter 9 Review: 16
   16. a. 0.23232323...
       b. the sum is $\frac{23}{99}$ (use geometric series formula).

5. Chapter 8 Review: 7, 11, 13, 19, 21
   These answers are all in the back of the book.

6. Show that the following integrals converge or diverge (you must actually show the comparison, not just give an idea of why you think it is based on looking at highest powers, etc):

a) $\int_2^\infty \frac{1}{x+1+\sin x} dx$

$$\frac{1}{x+1+\sin x} \geq \frac{1}{x+2}$$

hence

$$\int_2^\infty \frac{1}{x+1+\sin x} dx \geq \int_2^\infty \frac{1}{x+2} dx$$

which diverges, hence this integral diverges.

b) $\int_1^\infty \frac{1}{x^{10}+2x} dx$

on $[1, \infty)$ we see that

$$\frac{1}{x^{10}+2x} \leq \frac{1}{x^{10}}$$

and since $\int_1^\infty \frac{1}{x^{10}} dx$ converge, this integral must converge.

c) $\int_0^2 \frac{1}{x^3+2} dx$

This integral is only improper at the cube root of -2, which is not in the interval. Hence this integral converges.

d) $\int_1^\infty e^{-x^2} dx$

On this interval

$$\begin{aligned} x^2 &\geq x \\ -x^2 &\leq -x \\ e^{-x^2} &\leq e^{-x} \end{aligned}$$

and since $\int_1^\infty e^{-x} dx$ converges, this integral converges.

7. Write an integral that represents the arclength of the following curves:

a) $y = \sin x$ between $x = 0$ and $x = 2\pi$.

$$\int_0^{2\pi} \sqrt{1 + \cos^2 x} dx$$

b) $y = x^3 + 1$ between $x = -1$ and $x = 1$.

$$\int_{-1}^{1} \sqrt{1 + 9x^4} dx$$

c) the parametric curve $x = e^{2t}$, $y = \cos t$ for $t \in [0, 3]$.

$$\int_0^3 \sqrt{4e^{4t} + \sin^2 t} dt$$

d) the ellipse $x = 2\cos t$, $y = 4\sin t$ for $t \in [0, 2\pi]$.

$$\int_0^{2\pi} \sqrt{4\sin^2 t + 16\cos^2 t} dt$$

e) $y = \ln x$ between $x = 1$ and $x = 2$.

$$\int_1^2 \sqrt{1 + \frac{1}{x^2}} dx$$

8. Let R be the region bounded by the curve $y = x^2$, the x-axis, and the lines $x = 1$ and $x = 2$. Compute the volumes of the following solids:

a) the solid defined by rotating R about the y-axis.

$$\pi \int_0^4 (16 - y)\, dy = 56\pi$$

b) the solid defined by rotating R about the x-axis.

$$\pi \int_1^2 x^4 dx = \frac{31}{5}\pi$$

c) the solid defined by rotating R about the line $y = 6$.

$$\pi \int_1^2 \left(36 - (6 - x)^2\right) dx = \frac{47}{3}\pi$$

d) the solid whose base is R and whose cross-sections perpendicular to the x-axis are circles.

$$\pi \int_1^2 \left(\frac{1}{2}x^2\right)^2 dx = \frac{31}{20}\pi$$

e) the solid whose base is R and whose cross-sections perpendicular to the y-axis are equilateral triangles.

$$\frac{\sqrt{3}}{2} \int_1^2 \left(x^2\right)^2 dx = \frac{31}{10}\sqrt{3}$$

9. Compute the following integrals:

a)

$$\int x^2 \ln x \; dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

b)

$$\int 2xe^{-x^2} dx = -e^{-x^2} + C$$

c)

$$\int \frac{1}{x^2 - 1} dx = \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C$$

d)

$$\int \frac{1}{x^2 + 1} dx = \arctan x + C$$

e)

$$\int \frac{x^2}{1 - x} dx = -x - \frac{1}{2}x^2 - \ln |x - 1| + C$$

f)

$$\int \left(x^2 + 1\right) \cos x \; dx = 2x \cos x - \sin x + x^2 \sin x + C$$

g)

$$\int \frac{x}{\sqrt{1 - 2x^2}} dx = -\frac{1}{2}\sqrt{1 - 2x^2} + C$$

h)

$$\int \frac{1}{\sqrt{1-2x^2}} dx = \frac{1}{2}\sqrt{2} \arcsin\left(x\sqrt{2}\right) + C$$

10. Suppose a chain is hanging over the side of a platform. If the chain is 3 m long and its mass is 6 kg/m, how much work is required to pull the chain up?

Ans: The work is

$$\int_0^3 6\,(9.8)\, y dy = 264.6 \text{ N.}$$

11. Consider a fish tank which is in the shape of a cube which is 4 ft on each side. Suppose the tank is completely full of water. How much pressure is on the bottom? How much force is exterted on each side? (Recall that the density of water is 62.4 lb/ft^3).

Ans: There is no force on the top. The force on the bottom is

$$(62.4)\,(4)^3 = 3993.6 \text{ lb.}$$

The force on each of the sides is

$$\int_0^4 (62.4)\,(4)\, y dy = 1996.8 \text{ lb.}$$

12. How much work does it take to pump out 28 cubic feet of water from the top of a rectangular container 15 feet high with a square base which is 2 feet by 2 feet (so the water is 7 feet high)?

Ans: The work is

$$\int_0^7 (2)\,(2)\,(15-h)\,(62.4)\, dh = 20093 \text{ ft-lb.}$$

13. Do the rest of the problems from the chapter 8 review sheet (you know you didn't do them all!)

(The answers are in a different document. Check website.)

14. Decide which of the following series converge and which diverge. Justify your answer:

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by integral comparison since $\int_1^{\infty} 2^{-x} dx$ converges.

$\sum_{n=1}^{\infty} \frac{1}{2^2}$ diverges since it is a multiple of $1+1+1+1+\cdots$

$\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$ converges by integral comparison since $\int_1^{\infty} \frac{1}{x^{1.1}} dx$ converges.

$\sum_{n=1}^{\infty} n^{-1}$ diverges (it is the harmonic series, so there is the argument we gave in class as well as the integral test since $\int_1^{\infty} \frac{1}{x} dx$ diverges).

$\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$ diverges by integral comparison since $\int_1^{\infty} \frac{1}{x^{0.2}} dx$ diverges.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by integral comparison since $\int_1^{\infty} \frac{1}{x^2} dx$ converges.

$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges by integral comparison, since $\int_1^{\infty} \frac{1}{x \ln x} dx = \int_0^{\infty} \frac{1}{u} du$ which diverges.

$\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges by integral comparison, since $\int_1^{\infty} e^{-x} dx$ converges.

15. True/False, if false correct:

a) All geometric series converge.

False. Geometric series (i.e. $\sum_{n=1}^{\infty} ar^n$) whose ratios (r) are less than 1 in absolute value (i.e. $|r| < 1$) converge.

b) For any p, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

False. They converge for $p > 1$.

c) The units kg/m^2 can denote pressure.

False. The units for pressure are N/m^2 (or lb/ft^2 in English units).

d) Force and pressure are the same thing.

False. Pressure is a force per unit area.

e) Mass and weight are the same thing.

False. Weight is a force gravity exerts on an object of a given mass.

f) Force and weight use the same units.

True. Weight is the force of gravity on an object, and hence both force and weight have units Newtons (N) or pounds (lb).

g) If $\int_1^{\infty} f(x)\, dx$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges.

True. This is the integral test.

h) The sequence $1, 1/2, 1/3, 1/4, 1/5, \ldots$ converges.

False. This is the harmonic series and we can see that it diverges by the integral test or by the argument given in class.

i) The series $\sum_{n=2}^{\infty} \frac{n^2-1}{n^2}$ converges.

False. We can do the integral test and see that $\int_2^{\infty} \frac{x^2-1}{x^2} dx = \int_2^{\infty} \left(1 - \frac{1}{x^2}\right) dx$ diverges.