Chapters 1-2-4: Ordinary Differential Equations

Sections 1.1, 1.7, 2.2, 2.6, 2.7, 4.2 & 4.3

1. Ordinary differential equations

- An ordinary differential equation of order n is an equation of the form
$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right). \qquad (1)$$
- A solution to this differential equation is an n-times differentiable function $y(x)$ which satisfies (1).
- **Example:** Consider the differential equation
$$y'' - 2y' + y = 0.$$
  - What is the order of this equation?
  - Are $y_1(x) = e^x$ and $y_2(x) = x\,e^x$ solutions of this differential equation?
  - Are $y_1(x)$ and $y_2(x)$ linearly independent?

Initial and boundary conditions

- An initial condition is the prescription of the values of y and of its $(n-1)$st derivatives at a point x_0,
$$y(x_0) = y_0,\ \frac{dy}{dx}(x_0) = y_1, \dots \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}, \qquad (2)$$
where y_0, y_1, ... y_{n-1} are given numbers.
- Boundary conditions prescribe the values of linear combinations of y and its derivatives for two different values of x.
- In MATH 254, you saw various methods to solve ordinary differential equations. Recall that initial or boundary conditions should be imposed after the general solution of a differential equation has been found.

2. Existence and uniqueness of solutions

- Equation (1) may be written as a first-order system
$$\frac{dY}{dx} = F(x, Y) \qquad (3)$$
by setting $Y = \left[y, \frac{dy}{dx}, \frac{d^2y}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right]^T$.
- Existence and uniqueness of solutions: if F in (3) is continuously differentiable in the rectangle
$$R = \{(x, Y),\ |x - x_0| < a,\ ||Y - Y_0|| < b,\ a, b > 0\},$$
then the initial value problem
$$\frac{dY}{dx} = F(x, Y), \qquad Y(x_0) = Y_0,$$
has a solution in a neighborhood of (x_0, Y_0). Moreover, this solution is unique.

Existence and uniqueness of solutions (continued)

- **Examples:**
  - Does the initial value problem
    $$y'' - 2y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$
    have a solution near $x = 0,\ y = 1,\ y' = 0$? If so, is it unique?
  - Does the initial value problem
    $$y' = \sqrt{y}, \qquad y(0) = y_0$$
    have a unique solution for all values of y_0?
  - Does the initial value problem
    $$y' = y^2, \qquad y(1) = 1$$
    have a solution near $x = 1$, $y = 1$? Does this solution exist for all values of x?

Existence and uniqueness for linear systems

- Consider a linear system of the form
  $$\frac{dY}{dx} = A(x)Y + B(x),$$
  where Y and $B(x)$ are $n \times 1$ column vectors, and $A(x)$ is an $n \times n$ matrix whose entries may depend on x.
- Existence and uniqueness of solutions: If the entries of the matrix $A(x)$ and of the vector $B(x)$ are continuous on some open interval I containing x_0, then the initial value problem
  $$\frac{dY}{dx} = A(x)Y + B(x), \qquad Y(x_0) = Y_0$$
  has a unique solution on I.

Existence and uniqueness for linear systems (continued)

- **Examples:**
  - Apply the above theorem to the initial value problem
    $$y'' - 2y' + y = 3x, \qquad y(0) = 1, \qquad y'(0) = 0$$
  - Does the initial value problem
    $$\begin{aligned} &y^{(4)} - x^3 y'' + 3y = 0, \\ &y(0) = 1,\ y'(0) = 1,\ y''(0) = 0,\ y^{(3)}(0) = 0 \end{aligned}$$
    have a unique solution on the interval $[-1, 1]$?

3. Linear differential equations and systems

- The general solution of a homogeneous linear equation of order n is a linear combination of n linearly independent solutions.
- As a consequence, if we have a method to find n linearly independent solutions, then we know the general solution.
- In MATH 254, you saw methods to find linearly independent solutions of homogeneous linear ordinary differential equations with constant coefficients.
- This includes linear equations of the form $ay'' + by' + cy = 0$, and linear systems of the form $\frac{dY}{dx} = A\,Y$, where A is an $n \times n$ constant matrix and $Y(x)$ is a column vector in $\mathbb{R}^n$.

Linear differential equations and systems (continued)

- A set $\{y_1(x), y_2(x), \cdots, y_n(x)\}$ of n functions is linearly independent if its Wronskian is different from zero.
- Similarly, a set of n vectors $\{Y_1(x), Y_2(x), \cdots, Y_n(x)\}$ in $\mathbb{R}^n$ is linearly independent if its Wronskian is different from zero.
- The Wronskian of n functions $y_1(x),\ y_2(x),\ \cdots,\ y_n(x)$ is given by

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Linear differential equations and systems (continued)

- The Wronskian of n vectors $Y_1(x),\ Y_2(x),\ \cdots,\ Y_n(x)$ in $\mathbb{R}^n$ is given by

$$W(Y_1, Y_2, \cdots, Y_n) = \det\left([Y_1\ Y_2\ \cdots\ Y_n]\right),$$

  where $[Y_1\ Y_2\ \cdots\ Y_n]$ denotes the $n \times n$ matrix whose columns are $Y_1(x),\ Y_2(x),\ \cdots,\ Y_n(x)$.
- Finding n linearly independent solutions to a homogeneous linear differential equation or system of order n, is equivalent to finding a basis for the set of solutions.
- The next two slides summarize how to find linearly independent solutions in two particular cases.

Homogeneous linear equations with constant coefficients

To find the general solution to an ordinary differential equation of the form $ay'' + by' + cy = 0$, where $a, b, c \in \mathbb{R}$, proceed as follows.

1. Find the characteristic equation, $a\lambda^2 + b\lambda + c = 0$ and solve for the roots λ_1 and λ_2.
2. If $b^2 - 4ac > 0$, then the two roots are real and the general solution is $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.
3. If $b^2 - 4ac < 0$ the two roots are complex conjugate of one another and the general solution is of the form $y = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$, where $\alpha = \Re e(\lambda_1) = \frac{-b}{2a}$, and $\beta = \Im m(\lambda_1) = \frac{\sqrt{4ac-b^2}}{2a}$.
4. If $b^2 - 4ac = 0$, then there is a double root $\lambda = -\frac{b}{2a}$, and the general solution is $y = (C_1 + C_2 x)\, e^{\lambda x}$.

Homogeneous linear systems with constant coefficients

To find the general solution of the linear system $\frac{dY}{dx} = A\,Y$, where A is an $n \times n$ matrix with constant coefficients, proceed as follows.

1. Find the eigenvalues and eigenvectors of A.
2. If the matrix has n linearly independent eigenvectors $U_1, U_2, \cdots, U_n$, associated with the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the general solution is
   $$Y = C_1 U_1 e^{\lambda_1 x} + C_2 U_2 e^{\lambda_2 x} + \cdots + C_n U_n e^{\lambda_n x},$$
   where the eigenvalues λ_i may not be distinct from one another, and the C_i's, λ_i's and U_i's may be complex.

If A has real coefficients, then the eigenvalues of A are either real or come in complex conjugate pairs. If $\lambda_i = \overline{\lambda_j}$, then the corresponding eigenvectors U_i and U_j are also complex conjugate of one another.

4. Nonhomogeneous linear equations and systems

- The general solution y to a non-homogeneous linear equation of order n is of the form

$$y(x) = y_h(x) + y_p(x),$$

  where $y_h(x)$ is the general solution to the corresponding homogeneous equation and $y_p(x)$ is a particular solution to the non-homogeneous equation.

- Similarly, the general solution Y to a linear system of equations $\frac{dY}{dx} = A(x)Y + B(x)$ is of the form

$$Y(x) = Y_h(x) + Y_p(x),$$

  where $Y_h(x)$ is the general solution to the homogeneous system $\frac{dY}{dx} = A(x)Y$ and $Y_p(x)$ is a particular solution to the non-homogeneous system.

Nonhomogeneous linear equations and systems (continued)

- In MATH 254, you saw methods to find particular solutions to non-homogeneous linear equations and systems of equations.
- You should review these methods and make sure you know how to apply them.