Chapter 13: Complex Numbers
Sections 13.5, 13.6 & 13.7
1. Complex exponential

The *exponential* of a complex number $z = x + iy$ is defined as

$$\exp(z) = \exp(x + iy) = \exp(x) \exp(iy)$$

$$= \exp(x) (\cos(y) + i \sin(y)) .$$
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- As for real numbers, the exponential function is equal to its derivative, i.e.

  $$\frac{d}{dz} \exp(z) = \exp(z). \quad (1)$$
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- The exponential is therefore entire.
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- The exponential is therefore **entire**.

- You may also use the notation \( \exp(z) = e^z \).
Properties of the exponential function

- The exponential function is periodic with period $2\pi i$: indeed, for any integer $k \in \mathbb{Z}$,

$$
\exp(z + 2k\pi i) = \exp(x) (\cos(y + 2k\pi) + i \sin(y + 2k\pi)) \\
= \exp(x) (\cos(y) + i \sin(y)) = \exp(z).
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Moreover,

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|\exp(z)| = |\exp(x)| \ |\exp(iy)| = \exp(x) \sqrt{\cos^2(y) + \sin^2(y)}
= \exp(x) = \exp(\Re(z)).
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$$|\exp(z)| = |\exp(x)| \cdot |\exp(iy)| = \exp(x) \sqrt{\cos^2(y) + \sin^2(y)}$$
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- As with real numbers,
  - $\exp(z_1 + z_2) = \exp(z_1) \exp(z_2)$;
  - $\exp(z) \neq 0$. 

Chapter 13: Complex Numbers
2. Trigonometric functions

The complex sine and cosine functions are defined in a way similar to their real counterparts,

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\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.
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- The tangent, cotangent, secant and cosecant are defined as usual. For instance,

\[
\tan(z) = \frac{\sin(z)}{\cos(z)}, \quad \sec(z) = \frac{1}{\cos(z)}, \quad \text{etc.}
\]
The rules of differentiation that you are familiar with still work.

**Example:**

Use the definitions of \( \cos(z) \) and \( \sin(z) \),

\[
\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.
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to find \((\cos(z))'\) and \((\sin(z))'\).
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- to find $(\cos(z))'$ and $(\sin(z))'$.

- Show that Euler’s formula also works if $\theta$ is complex.
3. Hyperbolic functions

The complex hyperbolic sine and cosine are defined in a way similar to their real counterparts,

\[
\cosh(z) = \frac{e^z + e^{-z}}{2}, \quad \sinh(z) = \frac{e^z - e^{-z}}{2}. \quad (3)
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3. Hyperbolic functions

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- The hyperbolic sine and cosine, as well as the sine and cosine, are **entire**.
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- The hyperbolic sine and cosine, as well as the sine and cosine, are entire.

- We have the following relations

\[
\cosh(iz) = \cos(z), \quad \sinh(iz) = i \sin(z),
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4. Complex logarithm

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Since the exponential is \( 2\pi i \)-periodic, the complex logarithm is multi-valued.

Solving the above equation for \( w = w_r + i w_i \) and \( z = re^{i\theta} \) gives

\[
e^{w} = e^{w_r} e^{i w_i} = re^{i\theta} \quad \Rightarrow \quad \begin{cases} e^{w_r} = r \\ w_i = \theta + 2p\pi \end{cases},
\]

which implies \( w_r = \ln(r) \) and \( w_i = \theta + 2p\pi, \quad p \in \mathbb{Z}. \)
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  \[ e^w = e^{w_r} e^{iw_i} = re^{i\theta} \quad \Rightarrow \quad \begin{cases} e^{w_r} = r \\ w_i = \theta + 2p\pi \end{cases}, \]
  which implies \( w_r = \ln(r) \) and \( w_i = \theta + 2p\pi, \ p \in \mathbb{Z} \).

- Therefore,
  \[ \ln(z) = \ln(|z|) + i \arg(z). \]
Principal value of \( \ln(z) \)

- We define the **principal value** of \( \ln(z) \), \( \text{Ln}(z) \), as the value of \( \ln(z) \) obtained with the principal value of \( \text{arg}(z) \), i.e.

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\text{Ln}(z) = \ln(|z|) + i \text{Arg}(z).
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\text{Ln}(z) = \ln(|z|) + i \arg(z).
\]

The negative real axis is called a branch cut of \( \text{Ln}(z) \).
Principal value of $\ln(z)$ (continued)

Recall that

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Since \( \text{arg}(z) = \text{Arg}(z) + 2p\pi, \; p \in \mathbb{Z}, \) we therefore see that \( \ln(z) \) is related to \( \text{Ln}(z) \) by

\[ \ln(z) = \text{Ln}(z) + i 2p\pi, \quad p \in \mathbb{Z}. \]
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**Examples:**
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**Examples:**

- \( \text{Ln}(2) = \ln(2) \), but \( \ln(2) = \ln(2) + i 2p\pi, \quad p \in \mathbb{Z}. \)
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\[ \text{Ln}(z) = \ln |z| + i \text{Arg}(z). \]

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**Examples:**
- \( \text{Ln}(2) = \ln(2) \), but \( \ln(2) = \ln(2) + i 2p\pi, \quad p \in \mathbb{Z} \).
- Find \( \text{Ln}(-4) \) and \( \ln(-4) \).
Recall that
\[ \text{Ln}(z) = \ln |z| + i \arg(z). \]

Since \( \arg(z) = \arg(z) + 2p\pi, \ p \in \mathbb{Z} \), we therefore see that \( \ln(z) \) is related to \( \text{Ln}(z) \) by
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**Examples:**
- \( \text{Ln}(2) = \ln(2) \), but \( \ln(2) = \ln(2) + i 2p\pi, \quad p \in \mathbb{Z} \).
- Find \( \text{Ln}(-4) \) and \( \ln(-4) \).
- Find \( \ln(10i) \).
Properties of the logarithm

- You have to be careful when you use identities like

\[ \ln(z_1 z_2) = \ln(z_1) + \ln(z_2), \quad \text{or} \quad \ln \left( \frac{z_1}{z_2} \right) = \ln(z_1) - \ln(z_2). \]

They are only true up to multiples of \(2\pi i\).
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They are only true up to multiples of \(2\pi i\).

- For instance, if \(z_1 = i = \exp(i\pi/2)\) and \(z_2 = -1 = \exp(i\pi)\),

\[ \ln(z_1) = i \frac{\pi}{2} + 2p_1 i\pi, \quad \ln(z_2) = i\pi + 2p_2 i\pi, \quad p_1, p_2 \in \mathbb{Z}, \]

and

\[ \ln(z_1 z_2) = i \frac{3\pi}{2} + 2p_3 i\pi, \quad p_3 \in \mathbb{Z}, \]

but \(p_3\) is not necessarily equal to \(p_1 + p_2\).
Properties of the logarithm (continued)

Moreover, with $z_1 = i = \exp(i\pi/2)$ and $z_2 = -1 = \exp(i\pi)$,

$$\text{Ln}(z_1) = i \frac{\pi}{2}, \quad \text{Ln}(z_2) = i \pi,$$

and

$$\text{Ln}(z_1 z_2) = -i \frac{\pi}{2} \neq \text{Ln}(z_1) + \text{Ln}(z_2).$$
Properties of the logarithm (continued)

Moreover, with \( z_1 = i = \exp(i\pi/2) \) and \( z_2 = -1 = \exp(i\pi) \),

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\ln(z_1) = i\frac{\pi}{2}, \quad \ln(z_2) = i\pi,
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and

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\ln(z_1 z_2) = -i\frac{\pi}{2} \neq \ln(z_1) + \ln(z_2).
\]

However, every branch of the logarithm (i.e. each expression of \( \ln(z) \) with a given value of \( p \in \mathbb{Z} \)) is analytic except at the branch point \( z = 0 \) and on the branch cut of \( \ln(z) \). In the domain of analyticity of \( \ln(z) \),

\[
\frac{d}{dz} (\ln(z)) = \frac{1}{z}.
\]
5. Complex power function

If $z \neq 0$ and $c$ are complex numbers, we define

$$z^c = \exp(c \ln(z)) = \exp(c \ln(z) + 2pc\pi i), \quad p \in \mathbb{Z}.$$
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$$z^c = \exp(c \ln(z))$$

$$= \exp(c \text{Ln}(z) + 2pc\pi i), \quad p \in \mathbb{Z}.$$ 

- For $c \in \mathbb{C}$, this is again a multi-valued function, and we define the principal value of $z^c$ as

$$z^c = \exp(c \text{Ln}(z))$$