# Chapters 7-8: Linear Algebra Sections 7.5, 7.8 & 8.1

Chapters 7-8: Linear Algebra

#### 1. Linear systems of equations

• A linear system of equations of the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
  
 $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$   
...

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

can be written in matrix form as AX = B, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

# Solution(s) of a linear system of equations

- Given a matrix A and a vector B, a solution of the system AX = B is a vector X which satisfies the equation AX = B.
- If B is not in the column space of A, then the system
   AX = B has no solution. One says that the system is not consistent. In the statements below, we assume that the system AX = B is consistent.
- If the null space of A is non-trivial, then the system AX = B has more than one solution.
- The system AX = B has a unique solution provided dim(N(A)) = 0.
- Since, by the rank theorem, rank(A) + dim(N(A)) = n (recall that n is the number of columns of A), the system
   AX = B has a unique solution if and only if rank(A) = n.

## Row operations.

- There are three types of row operations:
  - Multiply a nonzero constant times an entire row.  $(r_i \rightarrow ar_i)$
  - 2 Exchange rows.  $(r_i \rightarrow r_j \text{ and } r_j \rightarrow r_i)$
  - 3 Add a multiple of one row to another.  $(r_i \rightarrow ar_j + r_i)$
- Row operations do not change the span of the row space.
- There are corresponding column operations, which do not change the column space.

#### Row operations to solve linear systems.

Row operations can be used to solve a linear system AX = B

$$-x - 4y + z = 10$$
$$x + y - 2z = 2$$
$$2x - y - 5z = 3$$

• Write an augmented matrix (A|B).

• Use row operations to get zeroes in the first column:

$$\begin{pmatrix} -1 & -4 & 1 & | & 10 \\ 0 & -3 & -1 & | & 12 \\ 0 & -9 & -3 & | & 36 \end{pmatrix} \begin{array}{c} r_1 + r_2 \\ 2r_1 + r_3 \end{array}$$

Definitions Solutions

Do the same with the next column:

$$\left(\begin{array}{ccccc} -1 & -4 & 1 & | & 10 \\ 0 & -3 & -1 & | & 12 \\ 0 & 0 & 0 & | & 0 \end{array}\right) \quad -3r_2 + r_3$$

• This is equivalent to the simplified system

$$-x - 4y + z = 10$$
$$-3y - z = 12$$
$$0 = 0$$

• To solve the system, use back substitution.

# Row operations to compute the rank of a matrix.

- Given a matrix *A*, row operations do not change the row space.
- Since the matrix

$$A=\left(egin{array}{ccccc} -1 & -4 & 1 & 10\ 1 & 1 & -2 & 2\ 2 & -1 & -5 & 16 \end{array}
ight)$$

can be made into the matrix

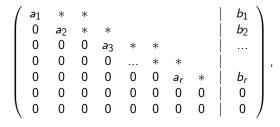
$$\mathcal{A}'=\left(egin{array}{ccccc} -1 & -4 & 1 & 10\ 0 & -3 & -1 & 12\ 0 & 0 & 0 & 0 \end{array}
ight)$$

by doing row operations, the two matrices have the same row spaces.

• It is easy to see that the first two rows are linearly independent, so the rank is 2.

# Consistency

- The system AX = B is consistent, i.e., has a solution if (equivalently):
  - Gaussian elimination on the augmented matrix (A|B) yields a matrix of the form:

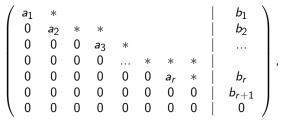


i.e., any rows reduced to all zeroes before the line are also zero after the line.

The rank of (A|B) is equal to the rank of A.

## Inconsistency

- The system AX = B is inconsistent, i.e., has NO SOLUTION if (equivalently):
  - Gaussian elimination on the augmented matrix (A|B) yields a matrix of the form:



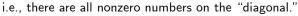
where  $b_{r+1} \neq 0$ , i.e., there is a row of zeroes before the line with a nonzero element after the line.

- The rank of (A|B) is greater than the rank of A.
- Solution The vector *B* is not in the column space of *A*.

# Unique solutions

- The system AX = B has one unique solution if (equivalently):
  - Gaussian elimination on the augmented matrix (A|B) yields a matrix of the form:

a <sub>1</sub>						$b_1$		
0	<b>a</b> 2					$b_2$		
0	0	<b>a</b> 3						
0	0	0						,
0	0	0	0	an		bn		
0	0	0	0	0		0		
0	0	0	0	0		0	Ϊ	
	0 0 0 0	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$



- 2 The rank of A is equal n (which is equal to the rank of (A|B)), which is the maximum rank (so it is essential that  $n \ge m$ ). This means that dim $(\mathcal{N}(A)) = 0$ , i.e., the nullspace is trivial.
- 3 The columns of A form a basis for the column space.

# Infinitely many solutions

- The system AX = B has lots of solutions if (equivalently):
  - Gaussian elimination on the augmented matrix (A|B) yields a matrix of the form:

,
)

i.e., there are zeroes on the diagonal and/or the last diagonal nonzero element is not next to the line  $\mid$  .

- The rank of A is less than n. This is equivalent to dim(N(A)) > 0.
- Interpretation of A do not form a basis of the column space.

Solution(s) of a linear system of equations (continued)

- A linear system of the form AX = 0 is said to be homogeneous.
- Solutions of AX = 0 are vectors in the null space of A.
- So If we know one solution  $X_0$  to AX = B, then all solutions to AX = B are of the form

$$X=X_0+X_h$$

where  $X_h$  is a solution to the associated homogeneous equation AX = 0.

In other words, the general solution to the linear system
 AX = B, if it exists, can be written as the sum of a particular solution X<sub>0</sub> to this system, plus the general solution of the associated homogeneous system.

# 2. Inverse of a matrix

 If A is a square n × n matrix, its inverse, if it exists, is the matrix, denoted by A<sup>-1</sup>, such that

$$A A^{-1} = A^{-1} A = I_n$$

where  $I_n$  is the  $n \times n$  identity matrix.

- A square matrix A is said to be singular if its inverse does not exist. Similarly, we say that A is non-singular or invertible if A has an inverse.
- The inverse of a square matrix  $A = [a_{ij}]$  is given by

$$A^{-1} = rac{1}{\det(A)} \left[ egin{array}{c} C_{ij} 
ight]^T$$
 ,

where det(A) is the determinant of A and  $C_{ij}$  is the matrix of cofactors of A.

Definitions Determinant of a matrix Properties of the inverse Linear systems of *n* equations with *n* unknowns

## Determinant of a matrix

• The determinant of a square  $n \times n$  matrix  $A = [a_{ij}]$  is the scalar

$$\det(A) = \sum_{i=1}^n a_{ij} C_{ij} = \sum_{j=1}^n a_{ij} C_{ij}$$

where the cofactor  $C_{ij}$  is given by

$$C_{ij}=\left(-1\right)^{i+j}\,M_{ij},$$

and the minor  $M_{ij}$  is the determinant of the matrix obtained from A by "deleting" the *i*-th row and *j*-th column of A.

• **Example:** Calculate the determinant of 
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
.

Definitions Determinant of a matrix Properties of the inverse Linear systems of *n* equations with *n* unknowns

### Properties of determinants

- If a determinant has a row or a column entirely made of zeros, then the determinant is equal to zero.
- The value of a determinant does not change if one replaces one row (resp. column) by itself plus a linear combination of other rows (resp. columns).
- If one interchanges 2 columns in a determinant, then the value of the determinant is multiplied by -1.
- If one multiplies a row (or a column) by a constant *C*, then the determinant is multiplied by *C*.
- If A is a square matrix, then A and A<sup>T</sup> have the same determinant.

Definitions Determinant of a matrix **Properties of the inverse** Linear systems of *n* equations with *n* unknowns

#### Properties of the inverse

• Since the inverse of a square matrix A is given by

$$A^{-1} = rac{1}{\det(A)} \left[ \mathit{C}_{ij} 
ight]^{\mathcal{T}}$$
 ,

we see that A is invertible if and only if  $det(A) \neq 0$ .

• If A is an invertible 2 × 2 matrix, 
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
, then
$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$
,

and  $det(A) = a_{11}a_{22} - a_{21}a_{12}$ .

• If A and B are invertible, then

$$(AB)^{-1} = B^{-1}A^{-1}$$
 and  $(A^{-1})^{-1} = A$ 

Linear systems of n equations with n unknowns

• Consider the following linear system of *n* equations with *n* unknowns,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
  

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
  
...  

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

- This system can be also be written in matrix form as AX = B, where A is a square matrix.
- If det(A) ≠ 0, then the above system has a unique solution X given by

$$X=A^{-1}B.$$

#### Linear systems of equations - summary

Consider the linear system AX = B where A is an  $m \times n$  matrix.

- The system may not be consistent, in which case it has no solution.
- To decide whether the system is consistent, check that *B* is in the column space of *A*.
- If the system is consistent, then
  - Either rank(A) = n (which also means that  $\dim(\mathcal{N}(A)) = 0$ ), and the system has a unique solution.
  - Or rank(A) < n (which also means that N(A) is non-trivial), and the system has an infinite number of solutions.

Linear systems of equations - summary (continued)

Consider the linear system AX = B where A is an  $m \times n$  matrix.

- If m = n and the system is consistent, then
  - Either det(A) ≠ 0, in which case rank(A) = n, dim(N(A)) = 0, and the system has a unique solution;
  - Or det(A) = 0, in which case dim(N(A)) > 0, rank(A) < n, and the system has an infinite number of solutions.</li>
- Note that when m = n, having det(A) = 0 means that the columns of A are linearly dependent.
- It also means that  $\mathcal{N}(A)$  is non-trivial and that rank(A) < n.

Eigenvalues Eigenvectors

# 3. Eigenvalues and eigenvectors

 Let A be a square n × n matrix. We say that X is an eigenvector of A with eigenvalue λ if

$$X \neq 0$$
 and  $AX = \lambda X$ .

• The above equation can be re-written as

$$(A-\lambda I_n)X=0.$$

- Since  $X \neq 0$ , this implies that  $A \lambda I_n$  is not invertible, i.e. that  $det(A \lambda I_n) = 0$ .
- The eigenvalues of A are therefore found by solving the characteristic equation  $det(A \lambda I_n) = 0$ .

# Eigenvalues

- The characteristic polynomial det(A λI<sub>n</sub>) is a polynomial of degree n in λ. It has n complex roots, which are not necessarily distinct from one another.
- If λ is a root of order k of the characteristic polynomial det(A – λI<sub>n</sub>), we say that λ is an eigenvalue of A of algebraic multiplicity k.
- If A has real entries, then its characteristic polynomial has real coefficients. As a consequence, if  $\lambda$  is an eigenvalue of A, so is  $\overline{\lambda}$ .
- It A is a 2 × 2 matrix, then its characteristic polynomial is of the form λ<sup>2</sup> - λ Tr(A) + det(A), where the trace of A, Tr(A), is the sum of the diagonal entries of A.

Eigenvalues Eigenvectors

# Eigenvalues (continued)

• Examples: Find the eigenvalues of the following matrices.

• 
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$
.  
•  $B = \begin{bmatrix} -1 & 9 \\ 0 & 5 \end{bmatrix}$ .  
•  $C = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}$ .  
•  $D = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$ .

Eigenvalues Eigenvectors

## Eigenvectors

• Once an eigenvalue  $\lambda$  of A has been found, one can find an associated eigenvector, by solving the linear system

$$(A-\lambda I_n)X=0.$$

- Since N(A λI<sub>n</sub>) is not trivial, there is an infinite number of solutions to the above equation. In particular, if X is an eigenvector of A with eigenvalue λ, so is αX, where α ∈ ℝ (or C) and α ≠ 0.
- The set of eigenvectors of A with eigenvalue λ, together with the zero vector, form a subspace of ℝ<sup>n</sup> (or ℂ<sup>n</sup>), E<sub>λ</sub>, called the eigenspace of A corresponding to the eigenvalue λ.
- The dimension of  $E_{\lambda}$  is called the geometric multiplicity of  $\lambda$ .

Eigenvalues Eigenvectors

## Eigenvectors (continued)

• **Examples:** Find the eigenvectors of the following matrices. Each time, give the algebraic and geometric multiplicities of the corresponding eigenvalues.

• 
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$
.  
•  $C = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}$ .  
•  $D = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$