Final Exam Problems

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & 1 \\ 3 & 7 & -2 \end{pmatrix}$$

a. Find all solutions to the equation $Ax = 0$.
Answer: Using row reduction we get

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & 1 \\ 3 & 7 & -2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 0 & 1 & 7 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

and so we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17t \\ -7t \\ t \end{pmatrix} = t \begin{pmatrix} 17 \\ -7 \\ 1 \end{pmatrix}$$

b. Find all solutions (if any) to $Ax = b$ where

$$b = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

Answer: Do the same with the augmented matrix

$$\begin{pmatrix} 1 & 2 & -3 & 2 \\ 2 & 5 & 1 & 0 \\ 3 & 7 & -2 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 7 & -4 \\ 0 & 1 & 7 & -4 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 7 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so there is a solution,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 17 \\ -7 \\ 1 \end{pmatrix}$$

c. Find all solutions (if any) to $Ax = b$ where

$$b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Answer: Do the same with the augmented matrix

$$\begin{pmatrix} 1 & 2 & -3 & 2 \\ 2 & 5 & 1 & 0 \\ 3 & 7 & -2 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 7 & -4 \\ 0 & 1 & 7 & -5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 7 & -4 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

so there are no solutions.

2.

a. Find all the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$.

Answer: The Eigenvalues are -1 and -6 corresponding to Eigenvectors $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ respectively.

b. Find the general solution of the homogeneous ODE

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Answer: Using part a, we see that the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

3.

a. Write the following differential equation as a first order system:

$$\frac{d^3y}{dx^3} + x^2 \frac{dy}{dx} - y^3 = 0.$$

Express your answer as $Y_0' = \cdots$, $Y_1' = \cdots$, etc.

Answer: We let $Y_0 = y$, $Y_1 = y'$, $Y_2 = y''$ and so

$$\begin{aligned} Y_0' &= Y_1 \\ Y_1' &= Y_2 \\ Y_2' &= -x^2 Y_1 + Y_0^3 \end{aligned}$$

b. For the following, circle ALL statements about the above differential equation which are necessarily true:

(i) The equation is homogeneous.

Answer: Yes, it is homogeneous. Each term has a y.

(ii) The equation is linear.

Answer: No, it is not linear because of the y^3 term.

4.

a. Find the Fourier series of the periodic function

$$f(x) = \begin{cases} 1 & \text{if } -1 < x \leq 0 \\ -2 & \text{if } 0 < x \leq 1 \end{cases}$$

extended to have period 2. It may help to know that $\cos(\pi n) = (-1)^n$ and $\sin(\pi n) = 0$ for any integer n.

Answer: We compute

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-1}^{1} f(x)\,dx = -1 \\ a_n &= \int_{-1}^{0} \cos(\pi n x)\,dx - 2\int_{0}^{1} \cos(\pi n x)\,dx = 0 \\ b_n &= \int_{-1}^{0} \sin(\pi n x)\,dx - 2\int_{0}^{1} \sin(\pi n x)\,dx \\ &= \frac{-1-(-1)^n}{n\pi} - 2\frac{(-1)^n - 1}{n\pi} = \frac{3}{n\pi}\left(-1-(-1)^n\right) \end{aligned}$$

So the series is

$$-1 + \sum_{n=1}^{\infty} \frac{3}{n\pi}\left(-1-(-1)^n\right)\sin n\pi x$$

b. At which points in the interval $-1 \leq x \leq 1$ does the series not converge to the value of the function? To which values does the series converge at these points?

Answer: At all of these points, the series converges to $\frac{1-2}{2} = -\frac{1}{2}$.

5.

a. Consider the following partial differential equation for $u(x,y)$:

$$\begin{aligned} u_{xx} + u_{yy} - 3u_y &= 0, \\ u(x,0) &= 0, \\ u(x,1) &= 0 \\ u(0,y) &= 0 \\ u(2,y) &= \sin(4\pi y). \end{aligned}$$

Use separation of variables to obtain two ODE associated with this PDE. (DO NOT solve them.)

Answer: We consider $u(x,y) = X(x)Y(y)$. The differential equation is

$$X''Y + XY'' - 2XY' = 0$$

so we get

$$\frac{X''}{X} = \frac{-Y'' + 2Y'}{Y} = k$$

for any constant k. This gives the two ODE

$$\begin{aligned} X'' &= kX \\ Y'' - 2Y' &= -kY. \end{aligned}$$

b. One (and only one) of those ODE's can be formed into a Sturm-Liouville equation with homogeneous boundary conditions using the boundary conditions from the PDE. State which one and give the boundary conditions.

Answer: The only one that gives a full set of boundary conditions is the one for Y, which gives $Y(0) = 0$ and $Y(1) = 1$. Note that you do get $X(0) = 0$ but cannot get another boundary condition for X.

6.

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for $u(x,t)$ on a finite domain $0 \le x \le 10$, with boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(10,t) &= 0 \end{aligned}$$

Recall that the general solution of the wave equation with these boundary conditions is of the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{10} + B_n \sin \frac{n\pi ct}{10} \right) \sin \frac{n\pi x}{10}.$$

If the equation is given initial conditions

$$u(x,0) = \sin(5\pi x),$$
$$\frac{\partial u}{\partial t}(x,0) = \sin(\pi x),$$

then find the particular solution for $u(x,t)$ (that is, find the coefficients A_n and B_n). Note: you may leave your answer with terms like $A_n = \sin\frac{3\pi n}{10} + \cos\pi n$ without further simplifying.

Answer: The two equations give

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{10} = \sin(5\pi x)$$
$$\sum_{n=1}^{\infty} B_n \frac{n\pi c}{10} \sin\frac{n\pi x}{10} = \sin(\pi x).$$

It follows that all coefficients are zero except $A_{50} = 1$ and $B_{10} = \frac{1}{\pi c}$. Hence the solution is

$$u(x,t) = \frac{1}{\pi c}\sin\pi ct\sin\pi x + \cos 5\pi ct\sin 5\pi x$$

7.

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = 0$$

on the WHOLE LINE ($x \in (-\infty,\infty)$). Find $\hat{u}(w,t)$, the Fourier transform of the solution $u(x,t)$. DO NOT solve for $u(x,t)$ (only its Fourier tranform). Note: the answer should contain $\hat{f}(w)$.

Answer: Taking the Fourier transform, we get

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -c^2 w^2 \hat{u}$$
$$\hat{u}(w,0) = \hat{f}(w)$$
$$\frac{\partial \hat{u}}{\partial t}(w,0) = 0.$$

The first equation can be solved as

$$\hat{u}(w,t) = A(w)\cos cwt + B(w)\sin cwt.$$

Pluggint in the initial conditions, we get

$$\hat{u}(w,t) = \hat{f}(w)\cos cwt.$$

8. For the following Sturm-Liouville problem, find all POSITIVE eigenvalues λ together with corresponding eigenfunctions.

$$y'' = -\lambda y$$
$$y'(0) = 0, \quad y'(2) = 0$$

Answer: The solutions are

$$y = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x.$$

Since r

$$y' = -a\sqrt{\lambda}\sin\sqrt{\lambda}x + b\sqrt{\lambda}\cos\sqrt{\lambda}x$$

we can use the initial conditions to conclude that

$$b = 0$$
$$\sin 2\sqrt{\lambda} = 0$$

and so

$$2\sqrt{\lambda} = \pi n$$
$$\lambda = \left(\frac{n\pi}{2}\right)^2$$

are the eigenvalues and the corresponding eigenfuctions are

$$y_n = \cos\frac{n\pi}{2}x.$$

9. Consider the heat equation

$$u_t = c^2 u_{xx}.$$

We want to solve this equation for $0 < x < 1$ and for all t with boundary conditions

$$u_x(0,t) = u_x(1,t) = 0$$

(**Notice the derivative in the boundary conditions**) and with the initial condition

$$u(x,0) = x.$$

a. Separate the variables $u(x,t) = F(x)G(t)$ and find 2 ordinary differential equations satisfied by F and G.

Answer: Using separation of variables, we get

$$FG' = c^2 F'' G$$

and so

$$\frac{G'}{c^2 G} = \frac{F''}{F}$$

and so this is equal to a constant and we get the two ODEs

$$\begin{aligned} F'' &= kF \\ G' &= c^2 k G. \end{aligned}$$

b. Using the boundary conditions, find the F_n.

Answer: The boundary conditions become

$$F'(0) = 0, \quad F'(1) = 0$$

and so we get

$$F = a \cos \sqrt{-k} x + b \sin \sqrt{-k} x$$

(positive k do not result in any solutions). The boundary conditions give $b = 0$ and $\sin \sqrt{-k} = 0$ and so $k = -(\pi n)^2$ for integers n. It follows that

$$F_n = \cos \pi n x.$$

(Note: $n = 0$ is allowed, too).

c. Find the G_n and write down the eigenfunctions u_n.

Answer:

$$G_n' = -c^2 \pi^2 n^2 G_n$$

and so

$$G_n = e^{-c^2 \pi^2 n^t t}.$$

d. Write down the general solution $u(x,t)$ and use the initial condition to find the coefficients.

Answer: The general solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-c^2 \pi^2 n^t t} \cos \pi n x.$$

The initial condition gives that

$$x = a_0 + \sum_{n=1}^{\infty} a_n \cos \pi n x.$$

We need to compute the coefficients of the even periodic extension, so we get

$$\begin{aligned} a_0 &= \int_0^1 x dx = \frac{1}{2} \\ a_n &= 2 \int_0^1 x \cos n \pi x dx = \frac{1}{\pi^2 n^2} (\cos \pi n - 1) \\ &= \frac{1}{\pi^2 n^2} \left((-1)^n - 1 \right) \end{aligned}$$

so the solution is

$$u\left(x,t\right)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\left(\left(-1\right)^{n}-1\right)}{\pi^{2}n^{2}}e^{-c^{2}\pi^{2}n^{t}t}\cos\pi nx.$$

e. What is $\lim_{t\to\infty} u(x,t)$

Answer: As $t \to 0$, all of the coefficients go to zero (because of the exponential), and so we get the limit is $\frac{1}{2}$.