# Math 323: Homework 10 Solutions

David Glickenstein

April 4, 2013

**8.9a)** The set of polynomials with integer coefficients is countable.

**Proof.** First consider the set $P_n$ of polynomials of degree $n$ with nonnegative integer coefficients. First, since there are infinitely many primes, there exists an injective map $p : \mathbb{N} \to Q$ which enumerates the set of prime numbers $Q \subset \mathbb{N}$. Then there is a map

$$f : P_n \to \mathbb{N}$$

given by

$$f\left(a_n x^n + \cdots + a_1 x + a_0\right) = 2^{a_0} 3^{a_1} 5^{a_2} \cdots p\left(n+1\right)^{a_n}.$$

By unique factorization, this map is one-to-one (the same argument as the proof that $A \times B$ is countable if $A$ and $B$ are countable – one could also show that $P_n$ is equinumerous to $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ where there are $n$ factors). Thus $P_n$ is countable. Finally, the set of polynomials $P$ can be expressed as

$$P = \bigcup_{n=0}^{\infty} P_n,$$

which is a union of countable sets, and hence countable. ∎

**8.9b)** The set of algebraic numbers is countable.

**Proof.** We know that the set $P$ of polynomials is countable. Each polynomial of degree $n$ has at most $n$ roots, thus for any polynomial $p$, the set $R_p$ of roots of $p$ is countable. Thus the set $A$ of algebraic numbers can be expressed as

$$A = \bigcup_{p \in P} R_p$$

is a countable union of countable sets, and hence countable. ∎

**8.9c)** Since the union of the set of transcendental numbers and the set of algebraic numbers is the set of real numbers, and the set of real numbers is uncountable, we must have that the set of transcendental numbers is uncountable (since the union of two countable sets is countable). Thus there are more transcendental numbers (uncountably many) than algebraic numbers (countably many).

**8.25)** Given two cardinal numbers $\alpha$ and $\beta$, we define the cardinal sum $\alpha + \beta$ as the cardinality of the set $A \cup B$, where the cardinality of $A$ is $\alpha$, the cardinality of $B$ is $\beta$, and $A \cap B = \varnothing$.

**Proposition 1** *The addition is well defined, i.e., if the cardinality of $A$ and $C$ is $\alpha$, the cardinality of $B$ and $D$ is $\beta$, and $A \cap B = \varnothing$ and $C \cap D = \varnothing$, then there is a bijection between $A \cup B$ and $C \cup D$.*

**Proof.** Given the circumstances, we know there are bijections $f : A \to C$ and $g : B \to D$ (since the cooresponding sets are equinumerous). We can now define a function $h : A \cup B \to C \cup D$ by

$$h\left(x\right) = \begin{cases} f\left(x\right) & \text{if } x \in A \\ g\left(x\right) & \text{if } x \in B. \end{cases}$$

This function is well-defined because $A \cap B = \varnothing$. Since $C \cap D = \varnothing$, we see that the preimage of element must be in $A$ or $B$ but not both, and since $f$ and $g$ are injections, that means that there is only one element, so $h$ is injective. Clearly $h$ is surjective, so it is a bijection. ∎

**Proposition 2** *We have* $\alpha + \beta = \beta + \alpha$ *and* $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

**Proof.** Since $\alpha + \beta$ is the cardinality of $A \cup B$ for some appropriate sets $A$ and $B$, since $A \cup B = B \cup A$, we see that $\alpha + \beta = \beta + \alpha$. Similarly, we have that

$$A \cup (B \cup C) = (A \cup B) \cup C,$$

and so the second equality follows. ■

**Proposition 3** $n + \aleph_0 = \aleph_0$ *for any finite cardinal* $n$.

**Proof.** We represent $\aleph_0$ by $\mathbb{N}$ and $n + \aleph_0$ by $\mathbb{N} \cup \{-(n-1), \cdots, -1, 0\}$, then the map

$$f : \mathbb{N} \to \mathbb{N} \cup \{-(n-1), \cdots, -1, 0\}$$

given by $f(m) = m - n$ is a bijection. ■

**Proposition 4** $\aleph_0 + \aleph_0 = \aleph_0$.

**Proof.** This can be seen, for instance, by seeing that $\mathbb{Z}_+$, $\mathbb{Z}_-$, and $\mathbb{Z} \setminus \{0\}$ are all equinumerous to $\mathbb{N}$, all of which we have shown. ■

**Proposition 5** $\aleph_0 + c = c$

**Proof.** We can represent $c$ as $(0, \infty)$ and $\aleph_0$ as $\{0, -1, -2, \ldots\}$, and then use the bijection between $\mathbb{N}$ and $\mathbb{Z}$ to replace $\mathbb{N} \subseteq (0, \infty)$ with $\mathbb{Z}$ and leave everything else alone, giving a bijection $(0, \infty) \cup \{0, -1, -2, \ldots\} \to (0, \infty)$. ■

**Proposition 6** $c + c = c$.

**Proof.** We can easily find a bijection $(0, \infty) \cup (-\infty, 0)$ and $\mathbb{R}$, which proves the proposition. ■