Math 443/543 Graph Theory Notes 11: Graph minors and Kuratowski's Theorem

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1 Graph minors

Let's revisit some definitions. Let $G = (V, E)$ be a graph.

Definition 1 Removing a vertex *means removing that vertex from the vertex set of* G *and removing all edges incident with that vertex from the edge set. We denote the graph obtained from* G *by removing a vertex* v *by* $G - v$.

Definition 2 Removing an edge *means removing that edge from the edge set. We denote the graph obtained from* G *by removing an edge* uv *by* $G - uv$.

Definition 3 *S*uppressing a vertex of degree two *means removing that vertex and replacing the two edges incident on that vertex by a single edge.*

Definition 4 Contracting an edge $e = uv$ *means removing* u *and* v *from the vertex set and replacing it by a new vertex* z *and edges such that* z *is adjacent to all vertices which were adjacent to* u *or* v.

We can now use these to define induced graphs such as subgraphs as follows. Let H and G be two graphs.

Definition 5 H *is an* induced subgraph *of* G *if* H *can be obtained from* G *by a sequence of vertex removals. We write* $H \leq_I G$.

Definition 6 H *is a* subgraph *of* G *if* H *can be obtained from* G *by a sequence of vertex and edge removals. We write* $H \leq_S G$.

Definition 7 H *is a* topological minor *of* G *(also called subdivision or topological subgraph) if* H *can be obtained from* G *by a sequence of vertex removals, edge removals, and suppressions of vertices of degree two. We write* $H \leq_T G$.

Definition 8 H *is a* minor *of* G *if* H *can be obtained from* G *by a sequence of vertex removals, edge removals, suppression of vertices of degree two, and edge contractions. We write* $H \leq_M G$.

Remark 9 *The sequence could have length zero, so a graph is always an induced subgraph, subgraph, topological minor, and minor of itself.*

Remark 10 *There is a hierarchy, since $H \leq_I G$ implies that $H \leq_S G$ which implies that $H \leq_T G$ which implies that $H \leq_M G$.*

2 Ordering properties

Definition 11 *A relation $\leq$ on a set X is called a* quasi-ordering *if it is reflexive ($x \leq x$ for all $x \in X$) and transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in X$). We use $x \geq y$ to denote $y \leq x$ and $y < x$ to denote that $y \leq x$ and $y \neq x$.*

We think of an ordering as allowing us to compare any two things, but that is not necessarily true.

Examples: We see easily that $\leq_I, \leq_S, \leq_T, \leq_M$ are all quasi-orderings on the set of all graphs.

Proposition 12 *If $\leq$ is a quasi-ordering on X such that there is no infinite decreasing sequence $x_1 > x_2 > x_3 > \cdots$, then for every subset $Y \subset X$ there is a set $M \subset Y$ such that*

1. *for any y there is an element $m \in M$ such that $m \leq y$ and*
2. *if $m, m' \in M$ and $m \neq m'$, then m and m' are incomparable (i.e., we have neither $m \leq m'$ nor $m' \leq m$).*

The set M is called the set of minimal elements *of Y.*

Proof. Omitted. It's not so easy. See Diestel if interested. ■

The existence of such a minimal set is goint to be the key. This motivates the following definitions.

Definition 13 *A quasi ordering is* without infinite decent *if there are no infinite strictly decreasing sequences $x_1 > x_2 > x_3 > \cdots$. An* antichain *$A \subset X$ is a subset such that no two elements in A are comparable.*

Note that the previous proposition says for a quasi-ordering without infinite decent, there is an antichain M with the property 1.

Definition 14 *Let P be a property defined on the elements on a set X. We say that P is* closed *under $\leq$, or $\leq$-closed, if for every two elements $x, y \in X$, if x has property P and $y \leq x$, then y also has property P.*

Note that a property P on a set X determines a subset $X_P \subset X$ with the property that x has property P if and only if $x \in X_P$.

As an example. The property "is bipartite" is closed under the subgraph and induced subgraph ordering, but not under the topological minor or minor ordering. The property "is planar" is closed under all of these orderings.

Proposition 15 *Let P be $\leq$-closed, and let M be the set of minimal elements of the set $X_{P^c} = \{x \in X : x$ does note satisfy property $P\}$. Then x has property P if and only if there is no $m \in M$ with $m \leq x$.*

Proof. Suppose x does not have property P. Then $x \in X_{P^c}$, and since M is the set of minimal elements of X_{P^c}, there must be $m \in M$ with the property $m \leq x$. Now suppose that x satisfies property P. Then any m such that $m \leq x$ must also satisfy property P since P is $\leq$-closed Thus if $m \in M \subset X_{P^c}$, then we cannot have $m \leq x$. ■

Definition 16 *The minimal set M in the above proposition is called the* minimal forbidden set *of the property.*

We have seen this already with Kuratowski's theorem, where the minimal forbidden set of the property "is planar" on the set of graphs under the minor relation is $K_{3,3}$ and K_5.

Definition 17 *A relation $\leq$ on a set X is a* well-quasi-ordering *if for any infinite sequence $x_1, x_2, \ldots$ of elements of X, there are two indices $i < j$ such that $x_i \leq x_j$.*

Proposition 18 *$\leq$ is a well-quasi-ordering if and only if it is a quasi-ordering without infinite descent and infinite antichains.*

Proof. If $\leq$ is a well-quasi-ordering, then clearly there are no infinite antichains (otherwise there is an infinite sequence of incomparables) and it is without infinite descent (since an infinite decreasing sequence does not satisfy the ordering assumption. If $\leq$ is not a quasi-ordering, there must be an infinite sequence of elements which are incomparable or strictly decreasing. ■

Not a lot of quasi-orderings are well-quasi-ordering.

Example 19 *$\leq_I$ and $\leq_S$ are not well-quasi-orderings on the set of graphs. (Neither is $\leq_T$). Consider the sequence of cycles $C_3, C_4, C_5, \ldots$ of cyclic graphs. Each of these is incomparable to each other, so it is an infinite antichain.*

Theorem 20 (Kruskal, 1960) *The class of trees with the topological minor ordering is well-quasi-ordered.*

Theorem 21 (Robertson-Seymour, 1986-2004) *The class of finite graphs is well-quasi-ordered under the minor ordering. This means that for any infinite sequence of graphs $G_1, G_2, G_3, \ldots,$ there are $i < j$ such that G_i is a minor of G_j.*

This is nice and all, but why is this important?

Proposition 22 *If $\leq$ is a well-quasi-ordering and P is a $\leq$-closed property, then the minimal forbidden set of P is finite.*

Proof. We know that $\leq$ is without infinite descent, any subset has a minimal set, in particular X_{P^c} does. Furthermore, that minimal set is an antichain, and antichains must be finite, so the minimal forbidden set is finite. ■

Corollary 23 *If P is a minor-closed property of graphs, then there exists a finite collection of graphs $H_1, H_2, \ldots, H_k$ such that, for all graphs G, G has property P if and only if G has none of $H_1, \ldots, H_k$ as a minor.*

Corollary 24 *For every surface S there exists a finite set of graphs $H_1, \ldots, H_k$ so that a graph is embeddable in S if and only if G has none of the H_i as a minor.*

Note: we know this for the plane (and hence the sphere), but we do not know what these are for even the torus.

We remark that this says that if we can decide if a graph has a particular subgraph as a minor, then we can check to see if a graph has a property. This can be done in polynomial time.

Theorem 25 (Robertson-Seymour, 1995) *For a fixed graph H, there exists a polynomial time algorithm to decide if a given input graph has H as a minor or not.*

Corollary 26 *If P is a minor-closed property of graphs, then there exists a polynomial time algorithm to decide if a graph has property P.*

Note: this requires knowing the minimal forbidden set. This is quite difficult!

3 Proof of Kuratowski's theorem

This follows BM-9.4,9.5. We wish to prove that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. First, we need some new terminology.

3.1 Bridges

First we define a relation $\sim$. Let H be a subgraph of G. We define $\sim$ on $E(G) - E(H)$ by the condition that $e \sim e'$ if there exists a walk W such that the first and last edges are e and e', respectively, and no vertex of W is a vertex of H (except possibly the ends of the walk). This is an equivalence relation (i.e, symmetric, reflexive, transitive), so it partitions $E(G) - E(H)$.

Definition 27 *A subgraph of $G - E(H)$ is a* bridge *if it is induced by the equivalence relation $\sim$, i.e., if it consists of the set of all edges equivalent to a fixed edge $e \in E(G) - E(H)$.*

See Figure 9.9 in BM. It is easy to see that:

1. A bridge must be connected. (In fact, any two vertices in the bridge must be connected by a path which has vertices in H only, at most, at the ends.

2. Two bridges have no vertices in common except possibly vertices in H. In each bridge, any vertices in H are called *vertices of attachment.*

We will look at bridges associated to cycles. Henceforth, assume a bridge is a bridge of a cycle.

Definition 28 *A bridge with k vertices of attachment is called a* k-bridge*. Two k-bridges with the same vertices of attachment are called* equivalent k-bridges*.*

Definition 29 *The vertices of attachment of a k-bridge with $k \geq 2$ partition the cycle C into pieces called* segments*. Two bridges* avoid one another *if the vertices of attachment of one bridge lie entirely in one segment of the other bridge. Otherwise they* overlap*. Two bridges B, B' are* skew *if there are four distinct vertices u, v, u', v' of C such that u, v and u', v' are vertices of attachment of B and B' respectively and the vertices appear in a cyclic order u, u', v, v'.*

Again, see Figure 9.9 in BM, which shows some bridges which are skew and some which are not.

Theorem 30 *If two bridges overlap, then either they are skew or else they are equivalent* 3*-bridges.*

Proof. Suppose we have 2 bridges B, B' which overlap (and hence each have at least 2 vertices of attachment). Certainly, if either is a 2-bridge, then they must be skew (check it directly). Now assume that both have at least 3 vertices of attachment. If B and B' are equivalent and have more than 3 vertices of attachment, then they must be skew, so if they are equivalent, it must be a 3-bridge or skew. If they are not equivalent, then B' has a vertex of attachment u' between two consecutive vertices of attachment u and v of B. Since B and B' overlap, there must be a vertex of attachment v' of B' which does not lie in the segment of B containing u'. ∎

Theorem 31 *If a bridge B has three vertices of attachment v_1, v_2, v_3, then there exists a vertex v_0 in B which is not in C and three internally disjoint paths P_1, P_2, P_3 from v_0 to v_1, v_2, v_3 respectively.*

Proof. Let P be a v_1v_2-path in B which is internally disjoint from C. P must have an interior vertex v, otherwise the bridge is just P and would not contain v_3. Let Q be a v_3v-path in B, internally disjoint from C, and let v_0 be the first vertex of Q on P. We then define P_1, P_2, P_3 appropriately. ∎

We are primarily considering with bridges in plane graphs. If G is a plane graph and C is a cycle in G, then C is a Jordan curve and hence divides the plane into an interior region and an exterior region. Since a bridge is connected, it must lie entirely in one of these two regions.

Definition 32 *An* inner bridge *is one contained in the interior region. An* outer bridge *is one contained in the exterior region.*

Note the following.

Theorem 33 *Inner bridges avoid one another, as do outer bridges.*

Proof. Let B, B' be two inner bridges which overlap. Then by Theorem 30, they must be equivalent 3-bridges or skew. Suppose they are skew. Then there are vertices $u, v \in V(B)$ and $u', v' \in V(B')$ such that u, u', v, v' appear in cyclic order around the cycle C. Let P be a uv-path in B which is internally disjoint from C and let P' be a $u'v'$-path in B' internally disjoint from C. Since B and B' are different bridges, P and P' cannot have an interior vertex in common. But they also must both be part of the interior of C since they are both inner bridges. This is impossible by the Jordan curve theorem, considering the region defined by part of C and P, which must be a Jordan curve, and the path P' which must go from the inside of the region to the outside of the region.

Now suppose B and B' are equivalent 3-bridges. Let the common vertex set be $\{v_1, v_2, v_3\}$. By Theorem 31 there is a vertex v_0 in B and three paths P_1, P_2, P_3 in B as in the statement of the theorem. Similarly there is v_0 and P_1', P_2', P_3' in B'. P_1, P_2, P_3 divide the interior region of C into three regions, and v_0' must be in one of those regions. But there must also be a path P_i' which goes to a vertex v_i not in the region. This is impossible by the Jordan curve theorem.

The argument for outer bridges is similar. ■

The key fact is that certain bridges can be moved from the interior region of C to the exterior region of C.

Definition 34 *Let G be a plane graph. An inner bridge B of a cycle C in G is* transferable *if ithere exists a plane graph $\tilde{G}$ isomorphic (as graphs) to G which is identiacal to G except that B is not an outer bridge of C in $\tilde{G}$. We say $\tilde{G}$ is obtained from G by* transferring B.

Theorem 35 *An inner bridge that avoids every outer bridge is transferrable.*

Proof. Let G be an inner bridge that avoids every outer bridge. Then the vertices of attachment all lie on the boundary of ONE region of G contained in the exterior region of C (since every outer bridge is within a single segment of C formed by B. Thus B can then be redrawn in that region by switching the vertices in the interior to the exterior. ■

3.2 Proof

Recall that a cut vertex is a vertex v such that $G - v$ is disconnected. We can define a more general k-vertex cut.

Definition 36 *A k-*vertex cut *is a set of k vertices $\{v_1, v_2, \ldots, v_k\}$ such that $G - \{v_1, v_2, \ldots, v_k\}$ is disconnected. A graph is said to be k-connected if there are no ℓ-vertex cuts for $\ell < k$ (i.e., all vertex cuts must contain at least k vertices).*

For example, any connected graph is 1-connected. A cycle is 2-connected.

Now consider a graph G with a 2-vertex cut $\{u,v\}$. Then there exist subgraphs G_1, G_2 such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{u,v\}$. Suppose we join u and v by a new edge e to obtain graphs H_1 and H_2. Then $G = H_1 \cup H_2 - e$.

Lemma 37 *If G is nonplanar, then at least one of H_1 and H_2 is nonplanar.*

Proof. Suppose both were planar. Then there are plane graphs representing each, and one region R of the plane graph for H_1 has e as a boundary edge. The same is true for H_2, but one can make it so that e is on the exterior region. Then we can embed $H_1 \cup H_2$ by putting H_2 into the region R. Removing e would give a planar embedding for G. ■

Lemma 38 *Let G be a nonplanar connected graph that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. Then G is 3-connected and does not have multiple edges between the same vertices.*

Proof. Clearly it cannot have multiple edges, since if so, we could remove them. If G is not 3-connected, then there is a a 2-vertex cut $\{u,v\}$. Since G is nonplanar, we can construct H_1, H_2 as above, and at least one must be nonplanar, say H_1. We know that the order of H_1 is strictly less than the order of G, so it must contain a subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$. Since K is not a subgraph of G, it must contain the edge e. Let P be a uv-path in $H_2 - e$. Then G contains a subgraph $K \cup P - e$, which is a subdivision of K and hence a subdivision of K_5 or $K_{3,3}$, a contradition. ■

We are now ready for the proof. Let's start with a minimal nonplanar graph, i.e., one such that if we remove any edges it becomes planar. If we can find a subgraph isomorphic to a subdivision of $K_{3,3}$ or K_5 in this, then we can find one in any nonplanar graph. We first construct a canonical piece using bridges.

Claim 39 *There is an edge $uv \in E(G)$, such that $G - uv$ has a planar embedding with a cycle C containing u and v such that*

1. *C contains the maximal number of interior edges as possible,*
2. *there is an outer bridge of C consisting of one edge xy which is skew to uv, and*
3. *there is an inner bridge B of C which is skew to both uv and xy.*

Before we prove the claim, let's see how this implies the theorem. We divide into two cases: whether all vertices of attachment of B are among $\{x,y,u,v\}$, or if there is a vertex attachment which is not in that set.

Suppose all vertices of attachment of B are among $\{x,y,u,v\}$. Since B is skew to both xy and uv, B must have all four vertices of attachment. There must be a uv-path P and a xy-path Q in $G - uv$ which are internally disjoint from C and which intersect each other (exercise). First suppose they intersect in exactly one vertex. Then the graph consisting of uv, xy, P, Q, C form a subdivision of K_5 (See BM Figure 9.21). If they intersect in more than one vertex, let u' be the intersecting vertex closest to u on P and v' the intersecting vertex closest to v on P. This induces subpaths P_1 between u and u' and P_2 between v' and v. If we let Q_1 be the path between u' and v' in Q, we can construct a subdivision of $K_{3,3}$ using uv, xy, Q, P_1, P_2, and the segments (x, v) and(y, u) . See BM Figure 9.22.

Now suppose there is an additional vertex of attachment of B, $v_1 \notin \{u, v, x, y\}$. We may suppose v is in the segment (x, u) . First suppose that B has another vertex of attachment v_2 in the segment (y, v) . Then there must be a v_1v_2-path in B, and one has a subdivision of $K_{3,3}$ made from uv, xy, P, and C. (See BM Figure 9.19). Now suppose B has no vertex of attachment in the segment (y, v) . Since B is skew to xy and uv, B must have vertices of attachment v_2 in the segment $(u, y]$ (possibly including y) and v_3 in the segment $(x, v]$ (possibly including v). By Theorem 31, there must be a vertex and three internally disjoint paths P_1, P_2, P_3 which go to the three vertices of attachment. We then can see the subdivision of $K_{3,3}$ induced by xy, uv, P_1, P_2, P_3 and all the segment sof C except segement (v, y) . See BM Figure 9.20.

Proof of Claim 39. If G is nonplanar but does not contain a subdivision of K_5 or $K_{3,3}$, then G is 3-connected by Lemma 38. Thus $G - uv$ is 2-connected.Thus any two vertices lie on a cycle (if not, one could disconnect the two by removing a single edge, and the graph would not be 2-connected). Hence there is a cycle containing u and v in $G - uv$. Now take the cycle C containing the maximum number of interior edges.

Since $G - uv$ is 2-connected (and without multiple edges), each bridge must have at least 2 vertices of attachment. Now consider an outer bridge. If it is a k-bridge with $k > 2$, then one could make the cycle contain more edges in the interior, contradicting the fact that the cycle contains the maximal number of edges in the interior. Thus all outer bridges are 2-bridges. Furthermore, it can contain only one edge, since if there is another vertex in the bridge, then the two vertices of attachment form a cut set with 2 vertices, contradicting the fact that G is 3-connected.. Furthermore, the bridge must overlap uv, otherwise there is a cycle which contains more edges in the interior.

Finally, inner bridges avoid one another. Thus if

If no inner bridge skew to uv is skew to an outer bridge, then one inner bridge is transferable. But since all inner bridges avoid one another, every inner bridge skew to uv can be transferred. Thus there is an embedding of G, a contradiction. Thus there must be an inner bridge skew to uv and skew to an outer bridge. ■