Math 443/543 Graph Theory Notes 2: Transportation problems

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1 Königsberg bridge problem and Eulerian graphs

See the figure for the graph corresponding to the Königsberg bridge problem. The problem is whether one can traverse each edge exactly once.
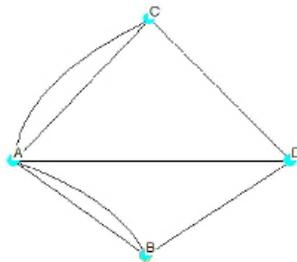
Here is the answer: No.

Here is the argument. We must start and stop at one of the vertices (possibly the same one). In particular, there must be at least one of the vertices B,C,D of degree three which is neither the start nor the stop point. Since it has degree three, the walker can enter along one edge (the walker does not start there) then leave by a second edge, then return by the last edge. Then the walker cannot move again. However, since this means the walker must stop there, we have a contradiction with the fact that we do not stop at that vertex.

Definition 1 *An* eulerian circuit *is a circuit containing all of the edges and vertices of the (multi-)graph. An* eulerian trail *is trail containing all of the edges and vertices of the (multi-)graph. A graph which contains an eulerian circuit is called an* eulerian graph *and a graph containing an eulerian trail is called a* traversable graph.

Recall that circuits and trails traverse each edge only once (but may traverse vertices any number of times).
We can characterize both eulerian circuits and trails:

Theorem 2 *A (multi-)graph is eulerian if and only if it is connected and every vertex has even degree. A (multi-)graph is traversable if and only if it is connected and exactly two vertices have odd degree.*

Proof. The two statements are proven almost the same way, so we only prove the first statement. If a graph G is eulerian, then it contains an eulerian circuit C which begins and ends at a vertex $v \in V(G)$. Since the circuit contains all vertices, there is a trail which connects any two vertices (a subset of the circuit C), and hence a path (by removing repeated occurrences of any vertices). Thus G is connected. Now consider a vertex $u \neq v$. Since the circuit neither begins nor ends at u, each time u is traversed, there must be an edge entering and an edge exiting. Thus the edges occur in pairs for each time the circuit visits u and u must have even degree. For the vertex v, every time it is visited other than the first and last time, there must be an edge entering and an edge exiting. Add to this one for the initial edge leaving and one for the last edge entering, v must have even degree too. (Note how this is changed for the second statement in the theorem).

For the converse, we must show that an eulerian circuit exists if the graph G is connected and every vertex has an even degree. Pick an initial vertex v and begin constructing a trail T until it can no longer be extended. The trail T must be a circuit (i.e., end at v) since if it ended at any other vertex, that vertex would have odd degree since it has entered and exited an even number of times and then finally entered with nowhere to go. Since all vertices have even degree, T is a circuit. Now we must extend T to be eulerian, i.e., to traverse all of the edges. Consider the graph G' gotten by removing the edges in the circuit T. Notice that G' also has the property that each vertex has even degree. If there are no edges in G', then T is eulerian. Otherwise, there must be an edge connected to a vertex v' which is covered by T (since G is connected). We may now construct a new circuit T' in the same way, beginning and ending at v'. Furthermore, we may append T' to the trail T to get a new circuit T'' which covers more edges. We may then continue the procedure of removing the edges of T'' in G until we get an eulerian circuit. We know this procedure will terminate since we are always adding edges to the trail T and their are a finite number of edges in G. ∎

Example 3.2 gives an interesting application of eulerian graphs.

2 Salesman problem and hamiltonian graphs

The traveling salesman problem is the following:

A salesman has to visit each city in his territory. The cities are connected by certain highways (or even by certain plane routes). Is it possible to plan a round trip so that he visits each city exactly once?

Definition 3 *A cycle containing all of the vertices of G is called a* hamiltonian cycle*. A graph containing a hamiltonian cycle is called a* hamiltonian graph.

It turns out that the problem of finding a hamiltonian cycle is NP-complete, which essentially means it is quite difficult. First, here is a condition satisfied by a hamiltonian graph:

Theorem 4 *If G is a hamiltonian graph, then for any nonempty, proper subset $S \subset V(G)$, $G - S$ has at most $|S|$ components.*

Proof. Let $\omega(G)$ denote the number of components of G. We know G contains a hamiltonian cycle C, and for a cycle, we have that

$$\omega(C - S) \leq |S|,$$

since removing one vertex leaves it connected, removing a second produces two components, removing a third produces another component, etc. However, since C is a subgraph of G which contains all of the vertices of G, we see that

$$\omega(G - S) \leq \omega(C - S)$$

(since $G - S = C - S$ union some edges).

Example: BM Figure 4.3 is not a hamiltonian graph. ∎

We give one sufficient condition for a graph to be hamiltonian.

Theorem 5 *Suppose G is a graph of order p such that $\deg v \geq p/2$ for every $v \in V(G)$. Then G is hamiltonian.*

Note that this is certainly not necessary, since we may consider a simple cycle, in which ever vertex has degree 2 but it is certainly hamiltonian.

Proof. If $p = 3$, then every vertex has degree larger than $3/2$, so G is the complete graph (a triangle) and that is hamiltonian. Now assume $p \geq 4$. Let $P = \{u_0, u_1, \ldots, u_k\}$ be a path which visits the most vertices (there may be several of these paths, but all visit the same number of vertices). We notice the following:

1. All vertices adjacent to v_0 must be in P, since otherwise we could make P larger. The same is true for v_k.

2. We must have $k \geq p/2 + 1$ since $\deg v_0 \geq p/2$.

3. There exists a number i, $1 \leq i \leq k$, such that u_0 is adjacent to u_i and u_k is adjacent to u_{i-1}. Suppose this were not the case, then for each u_i adjacent to u_0 (there are at least $p/2$ of these), there is another vertex not adjacent to u_k, which means there are $p/2 + 1$ vertices (including u_k) not adjacent to u_k. However, this is impossible since it would mean that $\deg u_k \leq p - (p/2 + 1) < p/2$.

4. Thus there is a cycle $C = \{u_0, u_i, u_{i+1}, \ldots, u_k, u_{i-1}, u_{i-2}, \ldots, u_0\}$.

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We claim that C is hamiltonian. Suppose that $v \in V(G)$ is not in C. We know that v is not connected to any vertex on C since if it were, then we would have a path longer than P (using the cycle C). Thus v is not connected to $1 + p/2$ of the vertices, which means that $\deg v \leq p - (p/2 + 1) < p/2$. A contradiction. ■

Notice that we only used that $\deg v + \deg v_0 \geq p$. Thus we may conclude:

Lemma 6 *Suppose G is a (p, q)-graph (NOT multi-graph) such and $u, v \in V(G)$ are nonadjacent and $\deg u + \deg v \geq p$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Proof (Sketch). Certainly if G is hamiltonian, so is $G + uv$. If $G + uv$ is hamiltonian, then we take a hamiltonian cycle C. If C does not contain uv, we are done. If C does, then there is a path P from $u = u_0$ to $v = u_k$ in G as in the proof of the theorem above. We can turn this into a cycle in the same way as above, and it must be hamiltonian. ■

This motivates the following definition:

Definition 7 *The closure of a (p, q)-graph G, denoted $c(G)$, is the graph obtained by recursively joining vertices $u, v \in V(G)$ which satisfy $\deg u + \deg v \geq p$ until all such pairs are joined by an edge.*

Proposition 8 *$c(G)$ is well defined, i.e., if we had joined vertices in another order, then we obtain the same graph.*

Proof. Let G_1 and G_2 be two graphs obtained from G as in the definition of the closure. Say G_1 is gotten by adding edges $\{e_1, e_2, \ldots, e_n\}$ in that order. Let $e_{k+1} = uv$ be the first such edge which is not in G_2 (if one exists). Then the graph $H = G + e_1 + \cdots + e_k$ must be a subgraph of G_2. By the way we defined G_1, we must have that

$$\deg_H u + \deg_H v \geq p.$$

However, since H is a subgraph of G_2, we must have

$$\deg_H u \leq \deg_{G_2} u$$
$$\deg_H v \leq \deg_{G_2} v$$

and thus

$$\deg_{G_2} u + \deg_{G_2} v \geq p.$$

However, this is a contradiction, since G_2 contains no such pairs of vertices. Thus every edge in G_1 is in G_2. A similar argument shows that G_2 is in G_1 and so $G_1 = G_2$. ■

We now get another condition for a graph to be hamiltonian.

Theorem 9 *A graph G is hamiltonian if and only if $c(G)$ is hamiltonian.*

Proof. Apply Lemma 6 each time an edge is added to construct the closure. ■

Corollary 10 *A graph with at least 3 vertices is hamiltonian if its closure is complete.*

See BM Figure 4.7, and show it is a hamiltonian graph.