1 Shortest path problems and Dijkstra’s algorithm

This section is from BM 1.8 (we will use $\phi$ instead of $w$). We consider the shortest path problem: Given a railway network connecting various towns, determine the shortest route between a given pair of towns.

**Definition 1** A network (or weighted graph) is a graph $G$ together with a map $\phi : E \rightarrow \mathbb{R}$. The function $\phi$ may represent the length of an edge, or conductivity, or cross-sectional area or many other things.

Given a network $(G, \phi)$, we can define the weight of a subgraph $H \subseteq G$ to be

$$\phi(H) = \sum_{e \in E(H)} \phi(e).$$

The problem is then: given two vertices $u_0, v_0 \in V(G)$, find a $u_0v_0$-path of smallest weight (where we consider the path as a subgraph).

**NOTE:** We will assume that $\phi(e) > 0$ for the remainder of the section, as this simplifies exposition.

Often, we will refer to the the weight of an edge as a length and the value of the smallest weight as the distance. We will present the algorithm of Dijkstra and Whiting-Hillier (found independently). In the sequel, we will assume that $\phi$ is defined on all pairs of vertices and $\phi(uv) = \infty$ if $uv \notin E(G)$.

**Definition 2** The distance between two vertices $u, v \in V(G)$ is equal to

$$d(u, v) = d_G(u, v) = \min \{ \phi(P) : P \text{ is a path from } u \text{ to } v \}.$$ 

A path $P$ which attains the minimum is called a shortest path.

We then have the following algorithm, known as Dijkstra’s algorithm:

1. Let $\ell(u_0) = 0$ and let $\ell(v) = \infty$ for all $v \neq u_0$. Let $S_0 = \{u_0\}$ and let

$$i = 0.$$
2. For each \( v \in S_i^r \), replace \( \ell (v) \) with
\[
\min_{u \in S_i} \{ \ell (v), \ell (u) + \phi(uv) \}.
\]

3. Compute \( M \) to be
\[
M = \min_{v \in S_i^r} \{ \ell (v) \}
\]
and let \( u_{i+1} \) be the vertex which attains \( M \).

4. Let \( S_{i+1} = S_i \cup \{ u_{i+1} \} \).

5. If \( i = p - 1 \), stop. If \( i < p - 1 \), then replace \( i \) with \( i + 1 \) and goto step 2.

**Lemma 3** If \( v_0, v_1, \ldots, v_k \) is a shortest path, then \( v_0, v_1, \ldots, v_j \) is a shortest path for any \( j \leq k \).

**Proof.** If there were a shorter path from \( v_0 \) to \( v_j \), then we could replace the current path with a shorter beginning and get a shorter path to \( v_k \).

Let’s prove that at the termination of the algorithm, \( \ell (u) = d (u, u_0) \). We will induct on \( i \). Clearly, this is true for \( i = 0 \). We will make the following inductive hypothesis:

- For every \( u \in S_i \), \( \ell (u) = d (u, u_0) \).

We have the base case, so we need only prove the inductive step. Suppose it is true for \( S_i \). We must show that
\[
d (u_0, u_{i+1}) = \ell (u_{i+1}).
\]

Let \( P = v_0, v_1, v_2, \cdot \cdot \cdot, v_k \), where \( v_0 = u_0 \) and \( v_k = u_{i+1} \), be a \( u_0u_{i+1} \)-path such that
\[
d (u_0, u_{i+1}) = \phi(P).
\]
If \( v_{k-1} \in S_i \), then the path \( P' = v_0, v_1, v_2, \ldots, v_{k-1} \) is a shortest path and by the inductive hypothesis \( \phi(P') = \ell (v_{k-1}) \). Thus
\[
d (u_0, u_{i+1}) = \phi(P) = \ell (v_{k-1}) + \phi(v_{k-1}u_{i+1}) \geq \ell (u_{i+1})
\]
but since \( d (u_0, u_{i+1}) \) is the minimum length path and \( \ell (u_{i+1}) \) is the length of some path, then we must have equality. Thus the inductive step is proven if \( v_{k-1} \in S_i \).

We now show \( v_{k-1} \in S_i \). Take the smallest \( j \) such that \( v_j \notin S_i \). Then since \( P_j = v_0, v_1, \ldots, v_j \) is a shortest path, we have, since \( v_{j-1} \in S_i \), that
\[
\ell (v_j) \leq \ell (v_{j-1}) + \phi(v_{j-1}v_j) = \phi(P_j) \leq \phi(P) \leq \ell (u_{i+1}).
\]
since \( \ell (u_{i+1}) = \min \{ \ell (u) : u \in S_i^r \} \), that means that all of the inequalities are equalities and \( j = k \) (since \( P_j = P \)) and \( v_{k-1} \in S_i \). By the previous argument, we are done.

See BM-1.8 for a discussion of the complexity of this algorithm. It turns out to be a good algorithm.