CHAPTER 3: TANGENT SPACE

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1. Tangent space

We shall define the tangent space in several ways. We first try gluing them together. We know vectors in a Euclidean space require a basepoint $x \in U \subset \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$. A $C^\infty$-manifold consists of a number of pieces of $\mathbb{R}^n$ glued together via coordinate charts, so we can define all tangents as follows. Consider what happens during a change of parametrization $\phi: V \to U$. It will take a vector $v$ to $d\phi(v)$. This motivates the following:

**Definition 1.** $T^{\text{glue}}M = \bigsqcup_i (U_i \times \mathbb{R}^n) / \sim$ where for $(x, v) \in U_i \times \mathbb{R}^n$, $(y, w) \in U_j \times \mathbb{R}^n$ we have $(x, v) \sim (y, w)$ if and only if $y = \phi_j \phi_i^{-1}(x)$ and $w = d(\phi_j \phi_i^{-1})_x(v)$.

The nice thing about this definition is it puts things together and gives the vectors in a good way. We define the tangent space at a point $p \in M$ as $T^{\text{glue}}pM = \{[p, v] : v \in \mathbb{R}^n\}$. It is easy to see that $T^{\text{glue}}M$ is an $n$-dimensional vector space. It is also easy to see that there is a map $\pi: T^{\text{glue}}M \to M$ defined by $\pi([p, v]) = p$ (since the parts of $M$ are really equivalence classes modulo equivalence. It also makes it clear that $T^{\text{glue}}M$ is a $C^\infty$ manifold.

We can define tangent spaces in two other ways.

**Definition 2.** $T^{\text{path}}pM = \{\text{paths } \gamma: (-\varepsilon, \varepsilon) \to M \text{ such that } \gamma(0) = p\} / \sim$ where $\alpha \sim \beta$ if $(\phi_i \circ \alpha)'(0) = (\phi_i \circ \beta)'(0)$ for every $i$ such that $p \in U_i$. $T^{\text{path}}M = \bigsqcup_{p \in M} T^{\text{path}}pM$.

This is a more geometric definition. Note that there is a map $\pi: T^{\text{path}}M \to M$ defined by $\pi(\gamma) = \gamma(0)$.

We shall show that $T^{\text{path}}M$ and $T^{\text{glue}}M$ are equivalent. The maps are

$\Phi: T^{\text{path}}pM \to T^{\text{glue}}pM$

defined by

$\Phi([\gamma]) = [\phi_i \circ \gamma(0), (\phi_i \circ \gamma)'(0)]$.

The inverse map is

$\Psi: T^{\text{glue}}pM \to T^{\text{path}}pM$

defined by

$\Psi([\phi_i(p), v]) = [t \to \phi_i^{-1}(\phi_i(p) + tv)]$.

It is clear that if well defined, they are inverses of each other. We need to show that $\Phi$ and $\Psi$ are well-defined. Clearly $\Phi$ is well defined because $\phi_i \circ \gamma(0) = [4.25]
\( \phi_i \circ \beta (0), (\phi_i \circ \gamma)' (0) = (\phi_i \circ \beta)' (0) \) for any \( \beta \in [\gamma] \). Also for any \( (\phi_j (p), w) \in [\phi_i (p), v] \) must satisfy \( d (\phi_i \circ \phi_j^{-1})_{\phi_i (p)} v = w \). Notice that

\[
\left\{ \phi_j \phi_i^{-1} (\phi_i (p) + tv) \right\}' (0) = d (\phi_j \circ \phi_i^{-1})_{\phi_i (p)} v = w = (\phi_j (p) + tw)' (0).
\]

The third way is in terms of germs of functions. A germ of a function is an equivalence class of functions.

**Definition 3.** Germs \( p \) is the set of functions \( f \in C^\infty (U_f) \) for \( p \in U_f \subset M \) modulo the equivalence that \([f] = [g] \) iff \( f (x) = g (x) \) for all \( x \in U_f \cap U_g \). Note that Germs \( p \) are an algebra since \([f] + [g] = [f + g] \) is well-defined, etc.

**Definition 4.** A derivation of germs is an \( \mathbb{R} \)-linear map \( X : \text{Germs}_p \rightarrow \mathbb{R} \) which satisfies

\[
X (fg) = f (p) X (g) + X (f) g (p).
\]

**Definition 5.** We define \( T_p^{der} M \) to be the set of derivations of germs at \( p \).

**Proposition 6.** Alternately, we may define the \( T_p^{der} M \) to be the set of derivations of smooth functions at \( p \).

**Proof.** Suppose \( X : C^\infty (M) \rightarrow \mathbb{R} \) is a derivation at \( p \). Then it determines a derivation of germs in the obvious way. Conversely, suppose \([f] \) is a germ at \( p \). Then there is a representative \( f : U \rightarrow \mathbb{R} \), and within that open set is a coordinate ball \( B \) centered at \( p \). Taking a smaller ball, we have a compact (closed) coordinate ball \( B' \) around \( p \) within the domain \( U \) of \( f \). We can consider the function \( x \rightarrow b (x) f (x) \), where \( b \) is a smooth bump function supported in \( U \) that is one on the ball \( B' \). These

This definition is nice because it shows how tangent vectors act on functions. We note derivations are a vector space since

\[
(X + Y) (fg) = X (f) g (p) + f (p) X (g) + Y (f) g (p) + f (p) Y (g)
= (X + Y) (f) g (p) + f (p) (X + Y) (g).
\]

A good example of a germ on \( U \subset \mathbb{R}^n \) is \( \frac{\partial}{\partial x^i} \bigg|_p \) since

\[
\frac{\partial}{\partial x^i} \bigg|_p (fg) = \frac{\partial f}{\partial x^i} (p) g (p) + f (p) \frac{\partial g}{\partial x^i} (p).
\]

These are linearly independent since \( \frac{\partial}{\partial x^i} \bigg|_p x^j = I^j_i \). We see that

\[
X (1) = 1 \cdot X (1) + X (1) \cdot 1
\]

so \( X (1) = 0 \). Similarly,

\[
X \left( (x^i - p^i) (x^j - p^j) \right) = 0.
\]

So by Taylor series:

\[
f (x) = f (p) + \frac{\partial f}{\partial x^i} \bigg|_p (x^i - p^i) + O \left( |x - p|^2 \right).
\]
We have formally that \( \frac{\partial}{\partial x^i} \big|_p \) span \( T^\text{der}_p U \). To make this argument rigorous, we know that

\[
f(x) = f(p) + \int_0^1 \frac{df}{dx^i}(tx + (1-t)p)\,dt
= f(p) + \int_0^1 \frac{\partial f}{\partial x^i} \big|_{tx+(1-t)p} (x^i - p^i)\,dt.
\]

Hence if we apply a derivation \( X \) we have

\[
X(f) = \int_0^1 \frac{\partial f}{\partial x^i} \big|_p \,dt \cdot X(x^i - p^i) + X \left( \int_0^1 \frac{\partial f}{\partial x^i} \big|_{tx+(1-t)p} \,dt \right) \cdot (p^i - p^i)
= \frac{\partial f}{\partial x^i} \big|_p \cdot X(x^i - p^i).
\]

Hence for \( U \subset \mathbb{R}^n \) we have a correspondence

\[
T^\text{der}_p U \to \mathbb{R}^n
\]

given by

\[
X \to (X(x^1 - p^1), \ldots, X(x^n - p^n))
\]

which is an invertible linear map with inverse

\[
\mathbb{R}^n \to T^\text{der}_p U
\]

\[
(s^1, \ldots, s^n) \to \left( X(f) = \frac{\partial f}{\partial x^i} \big|_p s^i \right).
\]

On a manifold, we define

\[
\frac{\partial}{\partial x^i} \big|_p f = \frac{\partial}{\partial x^i} \big|_{\phi_i(p)} (f \circ \phi_i)
\]

for coordinates \((x^1, \ldots, x^n) = \phi_i(p)\). Notice that under a change of coordinates from \((y^1, \ldots, y^n) = \phi_j(p)\) we have that

\[
\frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^k} \big|_{\phi_i(p)} (f \circ \phi_i)
= \frac{\partial}{\partial x^k} \big|_{\phi_j \circ \phi_i^{-1} \circ \phi_i(p)} (f \circ \phi_j \circ \phi_i^{-1} \circ \phi_i)
= \frac{\partial y^j}{\partial x^k} \big|_{\phi_i(p)} \frac{\partial}{\partial y^i} \big|_{\phi_j(p)} (f \circ \phi_j).
\]

Also, we have the projection \( \pi : T^\text{der} M \to M \).

**Proposition 7.** Let \( M = \mathbb{R}^n \). The derivations \( \frac{\partial}{\partial x^i} \big|_p \) form a basis for the derivations at \( p \).
Proof. We first see that \( X(c) = 0 \) if \( c \) is a constant function. By linearity of the derivation, we need only show that \( X(1) = 0 \). We compute:

\[
X(1) = X(1 \cdot 1) = 1 \cdot X(1) + X(1) \cdot 1 = 2X(1).
\]

We conclude that \( X(1) = 0 \).

Now, let \( X \) be a derivation and \( f \) a smooth function. We can write \( f \) as

\[
f(x) = f(p) + \int_0^1 \frac{\partial f}{\partial x^i} \bigg|_{tx+(1-t)p} (x^i - p^i) \, dt.
\]

By linearity and the derivation property, we have

\[
X(f) = X(f(p)) + X \left( \int_0^1 \frac{\partial f}{\partial x^i} \bigg|_{tx+(1-t)p} (x^i - p^i) \, dt \right).
\]

So, \( X(x^i - p^i) \) are just some numbers, and so we see that \( X \) is a linear combination of \( \frac{\partial}{\partial x^i} \mid_p \), meaning that these span the space of derivations! Since it is clear that \( \frac{\partial}{\partial x^i} \mid_p \) and \( \frac{\partial}{\partial x^j} \mid_p \) are linearly independent for each \( i \neq j \) (consider the functions \( x^i - p^i \)), the result follows.

\[\square\]

**Definition 8.** Given any smooth map \( F : M \to N \), there is a push forward \( F_* : T_pM \to T_{F(p)}M \) given as follows:

\[
F_*^{\text{path}}[\gamma] = [F \circ \gamma]
\]

\[
(F_*^{\text{der}} X) f = X(f \circ F).
\]

**Definition 9.** In any coordinate neighborhood \( (U, \phi) \) of \( p \), we define the derivation \( \frac{\partial}{\partial x^i} \mid_p \) by

\[
\frac{\partial}{\partial x^i} \bigg|_p = \phi_*^{-1} \frac{\partial}{\partial x^i} \bigg|_{\phi(p)}
\]

We may now see that \( T_p^{\text{der}} M \) is isomorphic to \( T_p^{\text{path}} M \). The map is

\[
[\gamma] \to \left\{ f \to \frac{d}{dt} \bigg|_{t=0} f(\gamma(t)) \right\}.
\]

We note that

\[
\frac{d}{dt} \bigg|_{t=0} f(\gamma(t)) = \frac{\partial (f \circ \phi_i^{-1})}{\partial x^j} \bigg|_{\phi_i \circ \gamma(0)} \cdot \frac{d(\phi_i \circ \gamma)^j}{dt} \bigg|_{t=0}
\]
and hence it is well-defined up to equivalence of paths. Note that \( \{ \phi^{-1}_i (p + t e_k) \}_{k=1}^n \) form a basis for \( \gamma \) and map to \( \frac{\partial}{\partial x^k} \bigg|_p \), so this is a linear isometry.

We will now use whichever definition we wish. Also note the following:

**Proposition 10.** If \( p \in U \subseteq M \) is an open set, then

\[ T_p M \cong T_p U. \]

Therefore, we will not make a distinction.

2. Computation in coordinates

Let’s compute the push-forward in coordinates. Recall that \( \left\{ \frac{\partial}{\partial x^k} \bigg|_p \right\}_{k=1}^m \) is a basis for \( T_p M \). Now, suppose that \( \left\{ \frac{\partial}{\partial y^a} \bigg|_{F(p)} \right\}_{a=1}^n \) is a basis for \( T_{F(p)} N \). Given a smooth map \( F : M \to N \), we should be able to compute the push forward in coordinates. If \( X \in T_p M \), we can write it in terms of the basis,

\[ X = X^k \frac{\partial}{\partial x^k} \bigg|_p \]

for some numbers \( X^k \in \mathbb{R} \). To compute the push forward, which is a linear map, we have that

\[ F_* X = X^k F_* \frac{\partial}{\partial x^k} \bigg|_p. \]

First, let’s suppose \( M = \mathbb{R}^m \) and \( N = \mathbb{R}^n \). To compute \( F_* \frac{\partial}{\partial x^k} \bigg|_p \), for \( f \in C^\infty (N) \) we need to compute

\[ \left( F_* \frac{\partial}{\partial x^k} \bigg|_p \right) f = \frac{\partial}{\partial x^k} \bigg|_p (f \circ F) \]

\[ = \frac{\partial f}{\partial y^a} \bigg|_{F(p)} \frac{\partial y^a}{\partial x^k} \bigg|_p \]

(note the summation) where, in the second expression, we really mean

\[ \frac{\partial y^a}{\partial x^k} \bigg|_p = \frac{\partial y^a (F(x))}{\partial x^k} \bigg|_p = \frac{\partial F^a}{\partial x^k} \bigg|_p \]

if \( F = (F^1, \ldots, F^n) \) is written in \( y \)-coordinates. Notice that once we have specified the coordinates, we have an expression for \( F_* \) in terms of the differential.

Now suppose we are on a manifold, then

\[ \left( F_* \frac{\partial}{\partial x^k} \bigg|_p \right) = \left( \psi_*^{-1} (\psi^* F_* \phi^{-1}) \phi_* \frac{\partial}{\partial x^k} \bigg|_\phi \right) \]

The middle map is known to us, as it is the differential of a map between \( \mathbb{R}^m \) and \( \mathbb{R}^n \), that is

\[ \psi_* F_* \phi^{-1} = \left( \frac{\partial F^a}{\partial x^k} (\phi (p)) \right)_{k=1, \ldots, m}^{a=1, \ldots, n} \]
where \( \hat{F} = \psi \circ F \circ \phi^{-1} \). In particular, we get

\[
\left( F_*, \frac{\partial}{\partial x^k} \big|_p \right) = \frac{\partial \hat{F}^a}{\partial y^a} \left( \phi(p) \right) \frac{\partial}{\partial y^a} \big|_{F(p)}
\]

One can also consider change of coordinates. If \((U, \phi)\) and \((V, \psi)\) are coordinate charts with coordinates \((x^i)\) and \((\tilde{x}^i)\), then any tangent vector can be written as

\[
X = X^i \frac{\partial}{\partial x^i} \big|_p = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i} \big|_p.
\]

How are \(X^i\) and \(\tilde{X}^i\) related? We can compute:

\[
\tilde{X}^i \frac{\partial}{\partial \tilde{x}^i} \big|_p = \tilde{X}^i \psi_*^{-1} \frac{\partial}{\partial x^i} \big|_{\psi(p)}
\]
\[
= \tilde{X}^i \phi_*^{-1} \phi_* \psi_*^{-1} \frac{\partial}{\partial x^i} \big|_{\psi(p)}
\]
\[
= \tilde{X}^i \phi_*^{-1} \left( \phi \circ \psi^{-1} \right)_* \frac{\partial}{\partial x^i} \big|_{\psi(p)}
\]
\[
= \tilde{X}^i \phi_*^{-1} \left[ \frac{\partial (\phi \circ \psi^{-1})^k}{\partial x^i} (\psi(p)) \frac{\partial}{\partial x^k} \big|_{\phi(p)} \right]
\]
\[
= \tilde{X}^i \frac{\partial (\phi \circ \psi^{-1})^k}{\partial \tilde{x}^i} (\psi(p)) \frac{\partial}{\partial x^k} \big|_p
\]

and so

\[
X^k = \tilde{X}^i \frac{\partial (\phi \circ \psi^{-1})^k}{\partial \tilde{x}^i} (\psi(p)).
\]

**Example 1.** Calculate the differential of the map \(F : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{C}P^1\).