Global Differential Geometry HW 5

Due November 10, 2011

1) Consider the problem of connecting three given points with three path segments meeting at one point. Show that to be a critical configuration for the length functional, one needs each path must be a geodesic (or a reparametrization of one) and the three paths must meet at 120 degree angles.

2) Lee, Problem 7-1

3) Show that any two-dimensional manifold only has one sectional curvature, say K. Show that the Ricci and scalar curvatures are

$$\begin{aligned} \mathrm{Rc}\left(X,Y\right) &= Kg\left(X,Y\right) \\ R &= 2K. \end{aligned}$$

4) In this problem, we give geometric interpretations of the Ricci and scalar curvatures

a) Let B be a symmetric bilinear form on an inner product space (V,g), i.e.

$$B\left(x,y\right) = B\left(y,x\right).$$

Consider an orthonormal basis $e_1,\ldots,e_n$ of V so that if $x = x^i e_i$ then

$$B\left(x,x\right) = \sum_{i=1}^{n} \lambda_i \left(x^i\right)^2$$

for some real λ_i. Show that

$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} B\left(x,x\right) dS^{n-1} = \frac{1}{n} \sum \lambda_i$$

where $S^{n-1} = \partial B^n$ is the unit sphere, dS^{n-1} is the standard measure on the unit sphere, and ω_{n-1} is the $(n-1)$-dimensional volume of S^{n-1}.

b) Show that the scalar curvature R satisfies

$$\frac{1}{n} R\left(p\right) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathrm{Rc}\left(X,X\right)\ dS^{n-1}\left(X\right)$$

where $S^{n-1} \subset T_pM$ is the unit sphere in T_pM.

c) Show that the Ricci curvature

$$\frac{1}{n-1} \operatorname{Rc}(X,X) = \frac{1}{\omega_{n-2}} \int_{S^{n-2}} K(X,Y) \; dS^{n-2}(Y)$$

where S^{n-2} is the unit sphere in T_pM orthogonal to X and K is the sectional curvature.