A monotonicity property for weighted Delaunay triangulations

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1 Introduction

In [21], Rippa proved a remarkable theorem about the optimality of Delaunay triangulations. He considered a quantity he called the roughness $R(f, \mathcal{T})$, which is a function of some values f on a collection of data points in the plane and a (two-dimensional) triangulation $\mathcal{T}$ of the data points. The roughness is defined as the Sobolev semi-norm

$$R(f, \mathcal{T}) = \sum_i \int_{T_i} \left[\left(\frac{\partial \phi_i}{\partial x} \right)^2 + \left(\frac{\partial \phi_i}{\partial y} \right)^2 \right] dxdy,$$

where ϕ_i is the linear interpolation of f over the triangle T_i in $\mathcal{T}$ and the sum is over all triangles in the triangulation. One may consider changing the triangulation by exchanging two triangles joined by an edge, forming a quadrilateral, by the triangles obtained by switching the diagonal of the quadrilateral; this is called an edge flip or a $2 \to 2$ bistellar flip. He showed that the roughness of a triangulation decreases when an edge is flipped to make the edge Delaunay. Since every triangulation can be transformed into a Delaunay triangulation by a sequence of edge flips, this implies that the roughness is minimized by a Delaunay triangulation.

The roughness is equal to the following functional, which we call the Dirichlet energy:

$$E(f, \mathcal{T}) = \frac{1}{4} \sum_{\{i,j\} \in \mathcal{T}} (\cot \alpha_{ij} + \cot \beta_{ij}) (f_j - f_i)^2,$$

where f is a function on the vertices of the triangulation $\mathcal{T}$, and α_{ij}, β_{ij} are the two angles opposite the edge $\{i, j\}$ (one of the terms is zero if $\{i, j\}$ is on the boundary). This formula is often called the cotan formula and goes at least as far back as Duffin's paper on lumped networks in 1959 [6]. It has been popularized in the last ten years or so by the work of Pinkall and Polthier [19]. Using the notation of the Dirichlet energy, Rippa's theorem may be stated as follows. The precise definitions of Delaunay triangulation and bistellar flip are giving in Section 2.

Theorem 1 ([21]) *Let $\mathcal{T}$ be a triangulation of a finite number of points in $\mathbb{R}^2$. Let $\mathcal{T}_0$ be the vertices of the triangulation and let $f : \mathcal{T}_0 \to \mathbb{R}$ be a function. Suppose $\mathcal{T}'$ is another triangulation of the same points which is gotten from $\mathcal{T}$ by a $2 \to 2$ bistellar flip of an edge e (in particular, $\mathcal{T}_0 = \mathcal{T}_0'$,) such that the hinge is Delaunay after the flip. Then*

$$E(f, \mathcal{T}') \leq E(f, \mathcal{T}),$$

where $E_{\mathcal{T}}$ and $E_{\mathcal{T}'}$ are the Dirichlet energies corresponding to $\mathcal{T}$ and $\mathcal{T}'$. As a consequence, the minimum is attained when all edges are Delaunay (and hence the triangulation is a Delaunay triangulation).

The key step in Rippa's proof is the exact calculation of $E(f, \mathcal{T}') - E(f, \mathcal{T})$. See also Powar's method of calculating this quantity [20].

In this paper we generalize Rippa's monotonicity lemma to the class of weighted Delaunay triangulations, sometimes referred to as regular triangulations or coherent triangulations in the literature (see, for instance, [1], [7], and [9, Chapter 7]). The proof we provide has the advantage of being a straightforward calculation, but the disadvantage that this calculation is quite long and complicated. It is an interesting observation that the Dirichlet energy defined for weighted triangulations does not appear to come from a piecewise linear interpolation except in the case of equal weights, when its formula is the cotan formula. The formula is derived in parallel to the classical Dirichlet energy instead of as an approximation to it.

The organization of the paper is as follows. We begin by introducing weighted Delaunay triangulations and the notation used in the rest of the paper. We then state and prove the flip monotonicity result. Finally we comment about what would be needed for a global theorem similar to Rippa's.

2 Weighted Delaunay triangulations

In this section we fix notation and review the definitions of weighted Delaunay triangulations. We denote simplices with braces $\{\}$ such as $\{i\}$ for a vertex, $\{i, j\}$ for an edge, and $\{i, j, k\}$ for a triangle. We shall often omit the braces for a vertex, denoting it simply as i. The entire triangulation is denoted $\mathcal{T} = \{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2\}$ where $\mathcal{T}_0$ are the vertices (0-dimensional simplices), $\mathcal{T}_1$ are the edges (1-dimensional simplices), and $\mathcal{T}_2$ are the triangles (2-dimensional simplices). In practice we may specify $\mathcal{T}$ by only specifying $\mathcal{T}_2$, where the other pieces can be seen as the vertices and edges in the triangles listed.

Two triangles adjacent to a common edge is called a *hinge*, and always has the form $\{\{i, j, k\}, \{i, j, \ell\}\}$. A triangulation can be altered by bistellar flips, the most important for our purposes being $2 \to 2$ bistellar flips, which we will call edge flips or just flips. A flip replaces the hinge $\{\{i, j, k\}, \{i, j, \ell\}\}$ by the hinge $\{\{i, k, \ell\}, \{j, k, \ell\}\}$. Such a flip can be seen in Figure 1.

We shall study the triangulations of a given set of points $\mathcal{T}_0$ in $\mathbb{R}^2$. The lengths of edges in the triangulation (gotten as the distance between the points

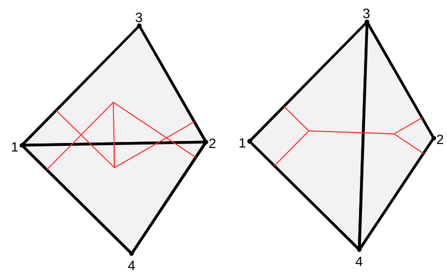

Figure 1: Hinges $\mathcal{T}$ and $\mathcal{T}'$ differing by a bistellar flip. The dual edges are also shown.

in $\mathbb{R}^2$) is given by a function

$$\ell : \mathcal{T}_1 \to (0, \infty),$$

where we write ℓ_{ij} as the length of edge $\{i, j\} \in \mathcal{T}_1$. The triangulation has the concepts of volume of a one-simplex (length) $|\{i, j\}| = \ell_{ij}$ and two-simplex (area of a Euclidean triangle) $|\{i, j, k\}|$. If a hinge is convex, then a flip makes sense and change the function ℓ, where the new length can be calculated as the length of the other diagonal.

We shall now add the weights to the structure.

Definition 2 *A* weighted triangulation $(\mathcal{T}, w)$ *is a triangulation* $\mathcal{T}$ *together with weights*

$$w : \mathcal{T}_0 \to \mathbb{R}.$$

We think of the weight w_i as the square of the radius of a circle centered at the vertex i. These weighted triangulations are used in the literature on regular triangulations such as [1] and [8]. Thinking of the weights in this way, in each triangle there exists a circle which is orthogonal to each of the circles centered at the vertices (this means they are perpendicular if they intersect, or else orthogonal in the sense described in [18, Section 40]). In this way, each triangle σ has a corresponding center $C(\sigma)$, which is the center of this circle, and the center has a weight $w_{C(\sigma)}$ which is the square of the radius of this circle. Figure 2 shows a triangle with weights w_i and circles centered at the vertices with radius $\sqrt{w_i}$ as well as the circle at the center of the triangle with radius $\sqrt{w_{C(\sigma)}}$, which is orthogonal to the other three circles. Note that since weights are only defined on vertices, flips are well defined operations on weighted triangulations. All of these ideas still make sense if the weights are negative if

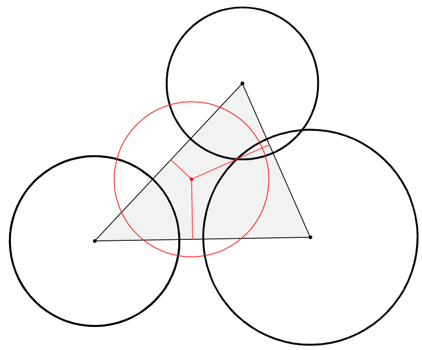

Figure 2: A weighted triangle with circles corresponding to the weights at the vertices. Central orthogonal circle and dual edges are also shown.

we use the formalism of [18, Section 40]. Note that since flips preserve the vertex set, they make sense on weighted triangulations. A flip will not change the set of vertex weights, although it will change the set of center weights.

An important particular case of weighted triangulations is when $w_i = 0$ for all vertices i. Delaunay triangulations are such (although Delaunay triangulations satisfy an extra condition, as noted below).

We now recall the definition of a weighted Delaunay triangulation (see, for instance, [1] or [7]). We first define centers which give geometric structure to the Poincaré dual of the triangulation. We will only need the (signed) length of edges $\bigstar\{i,j\}$ dual to edges $\{i,j\}$, although one can define signed volumes of all of the Poincaré duals of simplices (in any dimension) as is done in [12].

Let $d(x,p)$ be the Euclidean distance between points p and x. Define the power distance

$$\pi_p : \mathbb{R}^2 \rightarrow \mathbb{R}$$

by

$$\pi_p(x) = d(x,p)^2 - w_p \tag{1}$$

if p is a point weighted with w_p. The power is important as a function which is zero on the circle centered at p with radius $\sqrt{w_p}$, positive outside the circle, and negative inside the circle. Notice that if p is a vertex of a simplex σ and $c = C(\sigma)$ then $\pi_c(p) = w_p$ and $\pi_p(c) = w_c$, where the weight w_c is defined as the square of the radius of the orthogonal circle. If no orthogonal circle exists, a degenerate circle with negative weight can be found using the methods in [18, Section 40]. One may consider the space of circles in the plane as a vector space in 3-space given a particular indefinite inner product which generalizes the notion of two circles being orthogonal. In this way we have defined centers $C(\{i,j,k\})$ of triangles. Centers $C(\{i,j\})$ of edges are defined to be the points x on the line containing $\{i,j\}$ where $\pi_i(x) = \pi_j(x)$.

The distance from a triangle center $C(\{i,j,k\})$ to an edge center $C(\{i,j\})$ may be defined to be the positive distance from $C(\{i,j,k\})$ to $C(\{i,j\})$ if the center $C(\{i,j,k\})$ is on the same side of the line determined by $\{i,j\}$ as $\{i,j,k\}$ is and to be negative one times that distance if the center is on the opposite side from the triangle $\{i,j,k\}$. We denote by $h_{ij,k}$ this signed distance from $C(\{i,j,k\})$ to $C(\{i,j\})$, referring to the fact that $h_{ij,k}$ is like a height of $\{i,j\}$ inside $\{i,j,k\}$. Once can calculate $h_{ij,k}$ explicitly as follows. Use a Euclidean motion to move $\{i,j,k\}$ so that i is at the origin and $\{i,j\}$ is along the positive x-axis. Then $h_{ij,k}$ is the y-component of $C(\{i,j,k\})$. The components of $C(\{i,j,k\})$ can be found using the linear algebra of circles described in [18, Section 40], finding the circle which is orthogonal to the circles centered at the vertices of $\{i,j,k\}$ with radii $\sqrt{w_i}$, $\sqrt{w_j}$, $\sqrt{w_k}$. Simplifying the answer, one arrives at the following:

$$h_{ij,k} = \frac{\ell_{ij}}{2}\left(\cot\gamma_{kij} + \frac{w_i}{\ell_{ij}^2}\cot\gamma_{jik} + \frac{w_j}{\ell_{ij}^2}\cot\gamma_{ijk} - \frac{w_k}{2A_{ijk}}\right), \tag{2}$$

where $\ell_{ij} = |\{i,j\}|$ is the length, $A_{ijk} = |\{i,j,k\}|$ is the area, and γ_{ijk} is the interior angle at vertex i in triangle $\{i,j,k\}$. You can see the altitudes corresponding to $h_{ij,k}$ in Figures 1 and 2. The dual to edge $\{1,2\}$ in Figure 1 has negative length. Note that the signed distance is defined so that $\ell_{ij}h_{ij,k} + \ell_{ik}h_{ik,j} + \ell_{jk}h_{jk,i} = 2A_{ijk}$.

We define the Dirichlet energy of a function $f : \mathcal{T}_0 \to \mathbb{R}$ on a weighted triangulation of a hinge $\mathcal{T} = \{\{i,j,k\},\{i,j,\ell\}\}$ to be

$$\begin{aligned} E(f,\mathcal{T},w) = {} & \frac{h_{ij,k}+h_{ij,\ell}}{2\ell_{ij}}(f_j-f_i)^2 + \frac{h_{ik,j}}{2\ell_{ik}}(f_k-f_i)^2 + \frac{h_{jk,i}}{2\ell_{jk}}(f_j-f_k)^2 \\ & + \frac{h_{i\ell,j}}{2\ell_{i\ell}}(f_\ell-f_i)^2 + \frac{h_{j\ell,i}}{2\ell_{j\ell}}(f_j-f_\ell)^2 . \end{aligned}$$

For a general triangulation we can write the Dirichlet energy as

$$E(f,\mathcal{T},w) = \frac{1}{2}\sum_{\{i,j\}\in\mathcal{T}_1}\frac{|\bigstar\{i,j\}|}{|\{i,j\}|}(f_j-f_i)^2 \tag{3}$$

where $|\bigstar\{i,j\}|$ denotes the (signed) length of the edge dual to $\{i,j\}$, which is equal to $h_{ij,k}+h_{ij,\ell}$ if $\{i,j\}$ is an interior edge and $h_{ij,k}$ if $\{i,j\}$ is on the boundary. Notice that the dual edge $\bigstar\{i,j\}$ is orthogonal to the corresponding edge $\{i,j\}$.

The lengths $|\bigstar\{i,j\}|$ being positive is equivalent to the following classical definition of weighted Delaunay.

Definition 3 *An edge $\{i,j\}$ adjacent to the two triangles $\{i,j,k\}$ and $\{i,j,\ell\}$ is* weighted Delaunay *if $\pi_{C(\{i,j,k\})}(\ell) > w_\ell$ and $\pi_{C(\{i,j,\ell\})}(k) > w_k$, where $C(\{i,j,k\})$ is the center of $\{i,j,k\}$ and similarly for $\{i,j,\ell\}$. If the weights at the vertices are all equal to zero, a weighted Delaunay edge is said to be* Delaunay.

Sometimes we will instead say that the *hinge* is weighted Delaunay. A hinge is Delaunay if and only if it satisfies the local empty circumcircle property: the circle circumscribing $\{i, j, k\}$ does not contain ℓ. This is simply the interpretation of the definition when the weights are equal to zero. It is easy to see that an edge $\{i, j\}$ contained in a hinge $\{\{i, j, k\}, \{i, j, \ell\}\}$ is weighted Delaunay if and only if $|\bigstar \{i, j\}| \geq 0$. The proof is essentially the same as the corresponding theorem for Delaunay triangulations, which goes back at least to Rivin [22]. Details may be found in [11] or simply proved by looking at when the length of the dual edge is equal to zero. The formula for $h_{ij,k}$ makes it easy to see that the condition for being weighted Delaunay is unchanged by a weight scaling of the type $\{w_i \to w_i + c\}_{i \in \mathcal{T}_0}$, where c is some constant independent of i, since $h_{ij,k}$ is unaffected by such a deformation.

The formula for the Dirichlet energy is motivated as follows. We wish to write the Dirichlet energy as

$$E(f, \mathcal{T}, w) = -\frac{1}{2} \sum f_i \triangle f_i A_i$$

where $\triangle$ is a Laplacian operator and $A_i = |\bigstar i|$ is the area of the dual to the vertex i. We can write a general form for a Laplacian in terms of integration by parts

$$\int_U \triangle f dA = \int_{\partial U} \frac{\partial f}{\partial n} dS$$

where $\frac{\partial f}{\partial n}$ is the normal derivative and dS is the measure on the boundary. In our case we take $U = \bigstar i$ and this formula takes the form

$$\triangle f_i A_i = \sum_j \frac{|\bigstar \{i, j\}|}{|\{i, j\}|} (f_j - f_i)$$

if we choose the dual structure properly. This leads to the formula (3). This is the derivation of the Laplacian using Discrete Exterior Calculus as introduced in [13]. Special cases of Laplacians with this form have been studied, for instance, in [2] [3] [10] [11] [16] [17] [19]. These Laplacians are all Laplacians on the graph determined by the one-skeleton of the triangulation that the coefficients may be negative. In [6], Duffin interprets this type of Laplacian in terms of an electrical network on the one-skeleton of the triangulation. Each wire (edge) $\{i, j\}$ has a resistor with resistance $|\{i, j\}| / |\bigstar \{i, j\}|$. This make sense since the resistance of a wire should be proportional to its length and inversely proportional to its cross-sectional area. See [4] for general theory of graph Laplacians.

3 Monotonicity for weighted Delaunay triangulations

We now prove the main theorem for the Dirichlet energy on weighted triangulations.

Theorem 4 *Let* $(\mathcal{T}, w)$ *be a weighted triangulation of some points in* $\mathbb{R}^2$ *weights* w*. Let* $\mathcal{T}_0$ *be the vertices of the triangulation and let* $f : \mathcal{T}_0 \to \mathbb{R}$ *be a function. Suppose* $(\mathcal{T}', w)$ *is another weighted triangulation which is gotten from* $(\mathcal{T}, w)$ *by a* $2 \to 2$ *bistellar flip on edge e such that the hinge is weighted Delaunay after the flip. Then*

$$E(f, \mathcal{T}', w) \leq E(f, \mathcal{T}, w),$$

where $E(f, \mathcal{T}, w)$ *and* $E(f, \mathcal{T}', w)$ *are the Dirichlet energies of* f *corresponding to* $(\mathcal{T}, w)$ *and* $(\mathcal{T}', w)$*.*

The proof depends on the following generalization of Rippa's key lemma [21, Lemma 2.2] (see also [20]). Refer to Figure 1 for a picture of the hinges.

Lemma 5 *Let* $\mathcal{T} = \{\{1,2,3\}, \{1,2,4\}\}$ *and* $\mathcal{T}' = \{\{1,3,4\}, \{2,3,4\}\}$ *be two hinges differing by a flip along* $\{1,2\}$*. Then*

$$E(f, \mathcal{T}', w) - E(f, \mathcal{T}, w) = (f_{\mathcal{T}'}(c) - f_{\mathcal{T}}(c))^2 A_{1234}^2 \Phi$$

where

$$\Phi = \frac{2(r_3 r_4 - r_1 r_2) A_{1234} + w_1 A_{234} + w_2 A_{134} - w_3 A_{124} - w_4 A_{123}}{8 A_{123} A_{134} A_{234} A_{124}}, \tag{4}$$

A_{ijk} *is the area of* $\{i, j, k\}$*,*

$$A_{1234} = A_{123} + A_{124} = A_{134} + A_{234} \tag{5}$$

is the area of the hinge, c is the intersection of the diagonals, r_i *is the distance between c and vertex i, and* $f_{\mathcal{T}'}$ *and* $f_{\mathcal{T}}$ *are the piecewise linear interpolations of* f *with respect to the different triangulations. One can write*

$$f_{\mathcal{T}}(c) = \frac{r_1}{\ell_{12}} f_2 + \frac{r_2}{\ell_{12}} f_1$$
$$f_{\mathcal{T}'}(c) = \frac{r_3}{\ell_{34}} f_4 + \frac{r_4}{\ell_{34}} f_3.$$

The proof is somewhat involved, although straightforward. We use a proof which is more direct than the ones given by Rippa [21] and Powar [20] for the case of Delaunay triangulations.

Proof. Recall the definition of $h_{ij,k}$ from (2), which we think of as the height of the triangle $\{i, j, C(\{i, j, k\})\}$. For any function f, we can compute

$$E(f, \mathcal{T}', w) - E(f, \mathcal{T}, w) = \frac{1}{2} \sum_{i,j=1}^{4} a_{ij} f_i f_j,$$

where

$$a_{12} = \frac{h_{12,3}}{\ell_{12}} + \frac{h_{12,4}}{\ell_{12}}, \quad a_{13} = \frac{h_{13,2}}{\ell_{13}} - \frac{h_{13,4}}{\ell_{13}},$$
$$a_{14} = \frac{h_{14,2}}{\ell_{14}} - \frac{h_{14,3}}{\ell_{14}}, \quad a_{23} = \frac{h_{23,1}}{\ell_{23}} - \frac{h_{23,4}}{\ell_{23}},$$
$$a_{24} = \frac{h_{24,1}}{\ell_{24}} - \frac{h_{24,3}}{\ell_{24}}, \quad a_{34} = -\frac{h_{34,1}}{\ell_{34}} - \frac{h_{34,2}}{\ell_{34}},$$

and $a_{ii} = -\sum_{j\neq i} a_{ij}$ (where we have symmetrized $a_{ij} = a_{ji}$). We now wish to factor the coefficients.

We can easily figure out r_i in terms of areas in the following way. Let v_i be the point in the plane representing $\{i\}$. We see that $c = v_1 + \frac{r_1}{\ell_{12}}(v_2 - v_1) = v_3 + \frac{r_3}{\ell_{13}}(v_4 - v_3)$. By taking the cross product with $v_2 - v_1$ or $v_4 - v_3$ we find that

$$r_1 = \frac{\ell_{12} A_{134}}{A_{1234}} \quad \text{and} \quad r_3 = \frac{\ell_{34} A_{123}}{A_{1234}},$$

(recall the definition of A_{1234} from (5)) . Similarly,

$$r_2 = \frac{\ell_{12} A_{234}}{A_{1234}} \quad \text{and} \quad r_4 = \frac{\ell_{34} A_{124}}{A_{1234}}.$$

Thus

$$\begin{aligned} f_{\mathcal{T}'}(c) - f_{\mathcal{T}}(c) &= \frac{r_3}{\ell_{34}} f_4 + \frac{r_4}{\ell_{34}} f_3 - \frac{r_1}{\ell_{12}} f_2 - \frac{r_2}{\ell_{12}} f_1 \\ &= \frac{1}{A_{1234}} \left(A_{123} f_4 + A_{124} f_3 - A_{134} f_2 - A_{234} f_1 \right). \end{aligned}$$

Also useful will be the calculation

$$r_3 r_4 - r_1 r_2 = \frac{1}{A_{1234}^2} \left(\ell_{34}^2 A_{123} A_{124} - \ell_{12}^2 A_{234} A_{134} \right). \tag{6}$$

There are essentially two different types of coefficients to consider. We need only consider a_{12} and a_{13} since the others are similar. Let γ_{ijk} be the angle at vertex i in triangle $\{i, j, k\}$. Consider a_{12}.

$$\begin{aligned} a_{12} &= \frac{h_{12,3}}{\ell_{12}} + \frac{h_{12,4}}{\ell_{12}} \\ &= \frac{1}{2} \cot \gamma_{312} + \frac{w_1}{2\ell_{12}^2} \cot \gamma_{213} + \frac{w_2}{2\ell_{12}^2} \cot \gamma_{123} - \frac{w_3}{4A_{123}} \\ &\quad + \frac{1}{2} \cot \gamma_{412} + \frac{w_1}{2\ell_{12}^2} \cot \gamma_{214} + \frac{w_2}{2\ell_{12}^2} \cot \gamma_{124} - \frac{w_4}{4A_{124}} \\ &= \frac{1}{2} (\cot \gamma_{312} + \cot \gamma_{412}) + \frac{w_1}{2\ell_{12}^2} (\cot \gamma_{213} + \cot \gamma_{214}) \\ &\quad + \frac{w_2}{2\ell_{12}^2} (\cot \gamma_{123} + \cot \gamma_{124}) - \frac{w_3}{4A_{123}} - \frac{w_4}{4A_{124}}. \end{aligned} \tag{7}$$

Let θ be the angle at c in the triangle $\{1,3,c\}$. We shall use the fact that in any triangle $\{i,j,k\}$ we have $\ell_{ij} = \ell_{ik}\cos\gamma_{ijk} + \ell_{jk}\cos\gamma_{jik}$ to compute the parts.

$$\begin{aligned}
\cot\gamma_{312} + \cot\gamma_{412} &= \frac{\ell_{13}\ell_{23}\cos\gamma_{312}}{2A_{123}} + \frac{\ell_{14}\ell_{24}\cos\gamma_{412}}{2A_{124}} \\
&= \frac{\ell_{13}^2 - \ell_{12}\ell_{13}\cos\gamma_{123}}{2A_{123}} + \frac{\ell_{14}^2 - \ell_{12}\ell_{14}\cos\gamma_{124}}{2A_{124}} \\
&= \frac{\ell_{13}^2}{2A_{123}} + \frac{\ell_{14}^2}{2A_{124}} - \left(\left(\frac{\sin\gamma_{314}}{\sin\theta\sin\gamma_{123}} - \cot\theta\right) + \left(\frac{\sin\gamma_{413}}{\sin\theta\sin\gamma_{124}} + \cot\theta\right)\right) \\
&= \frac{\ell_{13}^2}{2A_{123}} + \frac{\ell_{14}^2}{2A_{124}} - \frac{1}{\sin\theta}\left(\frac{\sin\gamma_{314}}{\sin\gamma_{123}} + \frac{\sin\gamma_{413}}{\sin\gamma_{124}}\right) \\
&= \frac{\ell_{13}^2}{2A_{123}} + \frac{\ell_{14}^2}{2A_{124}} - \frac{1}{\sin\theta}\frac{\ell_{12}A_{134}A_{1234}}{\ell_{34}A_{123}A_{124}} \\
&= \frac{\ell_{13}^2A_{124} + \ell_{14}^2A_{123} - \ell_{12}^2A_{134}}{2A_{123}A_{124}} \\
&= \frac{\ell_{13}^2 + \ell_{14}^2}{2A_{1234}} + \frac{\ell_{13}^2A_{124}^2 + \ell_{14}^2A_{123}^2 - \ell_{12}^2A_{134}^2}{2A_{123}A_{124}A_{1234}} - \frac{\ell_{12}^2A_{134}A_{234}}{2A_{123}A_{124}A_{1234}}
\end{aligned}$$

since

$$\sin\gamma_{314} = \cos\gamma_{123}\sin\theta + \sin\gamma_{123}\cos\theta$$

and

$$\sin\gamma_{413} = \cos\gamma_{124}\sin\theta - \sin\gamma_{124}\cos\theta.$$

Furthermore,

$$\begin{aligned}
\ell_{13}^2A_{124}^2 + \ell_{14}^2A_{123}^2 - \ell_{12}^2A_{134}^2 &= \frac{1}{4}\ell_{12}^2\ell_{13}^2\ell_{14}^2\left(\sin^2\gamma_{124} + \sin^2\gamma_{123} - \sin^2\left(\gamma_{123}+\gamma_{124}\right)\right) \\
&= -\frac{1}{2}\ell_{12}^2\ell_{13}^2\ell_{14}^2\left(\sin\gamma_{123}\sin\gamma_{124}\cos\gamma_{134}\right) \\
&= -2A_{123}A_{124}\ell_{13}\ell_{14}\cos\gamma_{134}
\end{aligned}$$

since

$$\sin^2 A + \sin^2 B - \sin^2(A+B) = -2\sin A\sin B\cos(A+B).$$

Thus, using (6), we have

$$\begin{aligned}
\cot\gamma_{312} + \cot\gamma_{412} &= \frac{\left(\ell_{13}^2 + \ell_{14}^2 - 2\ell_{13}\ell_{14}\cos\gamma_{134}\right)}{2A_{1234}} - \frac{\ell_{12}^2A_{134}A_{234}}{2A_{123}A_{124}A_{1234}} \\
&= \frac{\ell_{34}^2A_{123}A_{124} - \ell_{12}^2A_{134}A_{234}}{2A_{1234}A_{123}A_{124}} \\
&= \frac{A_{1234}}{A_{123}A_{124}}\left(r_3r_4 - r_1r_2\right).
\end{aligned}$$

For the other parts,

$$\begin{aligned}
\cot\gamma_{213}+\cot\gamma_{214} &= \frac{\cos\gamma_{213}}{\sin\gamma_{213}}+\frac{\cos\gamma_{214}}{\sin\gamma_{214}}\\
&= \frac{\sin\gamma_{234}}{\sin\gamma_{213}\sin\gamma_{214}}\\
&= \frac{\ell_{12}^2 A_{234}}{2A_{123}A_{124}}
\end{aligned}$$

and

$$\cot\gamma_{123}+\cot\gamma_{124} = \frac{\ell_{12}^2 A_{134}}{2A_{123}A_{124}}.$$

Thus (7) implies that

$$\begin{aligned}
a_{12} &= \frac{A_{234}A_{134}}{4A_{123}A_{134}A_{234}A_{124}}\left(2A_{1234}\left(r_3r_4-r_1r_2\right)+w_1A_{234}+w_2A_{134}-w_3A_{124}-w_4A_{123}\right)\\
&= 2A_{234}A_{134}\Phi
\end{aligned}$$

using the definition of Φ in (4).

Now consider a_{13}. We can compute

$$\begin{aligned}
a_{13} &= \frac{h_{13,2}}{\ell_{13}}-\frac{h_{13,4}}{\ell_{13}}\\
&= \frac{1}{2}\cot\gamma_{213}+\frac{w_1}{2\ell_{13}^2}\cot\gamma_{312}+\frac{w_3}{2\ell_{13}^2}\cot\gamma_{123}-\frac{w_2}{4A_{123}}\\
&\quad -\left(\frac{1}{2}\cot\gamma_{413}+\frac{w_1}{2\ell_{13}^2}\cot\gamma_{314}+\frac{w_3}{2\ell_{13}^2}\cot\gamma_{134}-\frac{w_4}{4A_{134}}\right)\\
&= \frac{1}{2}\left(\cot\gamma_{213}-\cot\gamma_{413}\right)+\frac{w_1}{2\ell_{13}^2}\left(\cot\gamma_{312}-\cot\gamma_{314}\right)\\
&\quad +\frac{w_3}{2\ell_{13}^2}\left(\cot\gamma_{123}-\cot\gamma_{134}\right)-\frac{w_2}{4A_{123}}+\frac{w_4}{4A_{134}}.
\end{aligned}$$

We see that

$$\begin{aligned}
\cot\gamma_{213}-\cot\gamma_{413} &= \frac{\sin\gamma_{324}}{\sin\gamma_{213}\sin\theta}-\frac{\sin\gamma_{124}}{\sin\gamma_{413}\sin\theta}\\
&= \frac{\ell_{12}^2A_{134}A_{234}-\ell_{34}^2A_{123}A_{124}}{2A_{1234}A_{123}A_{134}}
\end{aligned}$$

since $\sin\gamma_{324}=-\cos\theta\sin\gamma_{213}+\sin\theta\cos\gamma_{213}$ and similarly $\sin\gamma_{124}=-\cos\theta\sin\gamma_{413}+\sin\theta\cos\gamma_{413}$. We also get

$$\begin{aligned}
\cot\gamma_{312}-\cot\gamma_{314} &= \frac{\cos\gamma_{312}\sin\gamma_{314}-\cos\gamma_{314}\sin\gamma_{312}}{\sin\gamma_{312}\sin\gamma_{314}}\\
&= -\frac{\sin\gamma_{324}}{\sin\gamma_{312}\sin\gamma_{314}}\\
&= -\frac{\ell_{13}^2A_{234}}{2A_{123}A_{134}}
\end{aligned}$$

and

$$\cot \gamma_{123} - \cot \gamma_{134} = \frac{\ell_{13}^2 A_{124}}{2A_{123}A_{134}}.$$

And so, using (6) and (4),

$$\begin{aligned} a_{13} &= \frac{-A_{234}A_{124}}{4A_{123}A_{134}A_{234}A_{124}} \left(2A_{1234}\left(r_3 r_4 - r_1 r_2\right) + w_1 A_{234} - w_3 A_{124} + w_2 A_{134} - w_4 A_{123}\right) \\ &= -2A_{234}A_{124}\Phi. \end{aligned}$$

A similar argument gives the other coefficients. Then we see, for instance, that

$$\begin{aligned} a_{11} &= -a_{12} - a_{13} - a_{14} \\ &= 2\left(-A_{234}A_{134} + A_{234}A_{124} + A_{234}A_{123}\right)\Phi \\ &= 2A_{234}^2\Phi \end{aligned}$$

with similar expressions for a_{22}, a_{33}, and a_{44}. Finally, we get that

$$E\left(f, \mathcal{T}', w\right) - E\left(f, \mathcal{T}, w\right) = \left(A_{123}f_4 + A_{124}f_3 - A_{234}f_1 - A_{134}f_2\right)^2 \Phi,$$

which is equivalent to the lemma. ■

Now we can prove the theorem.

Proof of Theorem 4. Since the coefficient $a_{12} = \frac{|\bigstar\{1,2\}|}{|\{1,2\}|}$ and $a_{34} = -\frac{|\bigstar\{3,4\}|}{|\{3,4\}|}$, we see that $a_{12} < 0$ and $a_{34} < 0$ if and only if $\mathcal{T}$ is not weighted Delaunay and $\mathcal{T}'$ is weighted Delaunay. Since all areas A_{ijk} are positive, $a_{12} < 0$ if and only if $\Phi < 0$ by formula (7) for a_{12}, and hence the result is proven. ■

Note that in the proof we have shown that $\Phi < 0$ if and only if $\mathcal{T}$ is not weighted Delaunay and $\mathcal{T}'$ is weighted Delaunay.

4 Remarks on a global theorem

Rippa's theorem (3) includes the implication that the Delaunay triangulation, a triangulation in which every edge is Delaunay, minimizes the Dirichlet energy. This follows immediately from the monotonicity under flips, since a Delaunay triangulation can be derived from any other triangulation by a sequence of flips. This "flip algorithm" was proposed by Lawson [15]. For proof that the flip algorithm finds the Delaunay triangulation, see [1] [7] [22].

In order to get the global statement for weighted Delaunay triangulations in the same way, one would need to know that a weighted Delaunay triangulation can be found using flips. This is not true in general (see [8, Section 5]). However, there are some conditions which ensure that the naive flip algorithm does work (see [12]), in which case one would conclude that the weighted Delaunay triangulation minimizes the Dirichlet energy, however these conditions are not particularly natural. In general, one may need to eliminate vertices which have empty dual cells in the weighted Delaunay triangulation (i.e., vertices i which have negative $|\bigstar i|$ with an appropriate definition of the dual area). One might try to eliminate vertices using $3 \to 1$ flips, which replace three triangles with a single triangle. However, even allowing such flips, we see by the example in [8, Section 5] that there are local minima which are not weighted Delaunay. In order to transform this example into a weighted Delaunay triangulation, one would have to flip weighted Delaunay edges incident on the inside triangles to become non-weighted Delaunay and then eliminate the vertices using a $3 \to 1$ flip. Hence in order to find the weighted Delaunay triangulation, we cannot only perform flips which decreases the Dirichlet energy.

Weighted Delaunay triangulations can be formed using the incremental algorithm in [8], which adds vertices incrementally only if the dual cell would have positive area, performing edge flips along the way in order to make the triangulation weighted Delaunay before adding the next vertex. It is shown that if one vertex is added to a weighted Delaunay triangulation, then a weighted Delaunay triangulation is reachable via edge flips. Theorem 4 says that once the vertex is added, the Dirichlet energy decreases until it reaches the weighted Delaunay triangulation. The addition of a new vertex may increase the Dirichlet energy.

5 Further remarks

It is an interesting fact that if the weights are all equal to zero, the Dirichlet energy (3) is always nonnegative because it is the restriction of the smooth Dirichlet energy

$$\int |\nabla f|^2 \, dA$$

to the space of piecewise-linear finite elements. It does not appear that the Dirichlet energy has this interpretation in the case of varying weights. This leads to the question of when (3) is necessarily positive. We can give a partial answer to this question by looking at the restriction to each triangle. On a triangle $\{1,2,3\}$, the Dirichlet energy is the quadratic form Q represented by the matrix $A = (a_{ij})_{i,j=1,2,3}$ where $a_{ij} = h_{ij,k}/\ell_{ij}$ if $i \neq j$ and $a_{ii} = -h_{ij,k}/\ell_{ij} - h_{ik,j}/\ell_{ik}$. It is easy to see that $(1,1,1)^T$ is an eigenvector with eigenvalue 0 for this matrix. It is sufficient to give conditions for when this matrix is rank 2. We find that the 2×2 minors all look like

$$\begin{aligned}
M &= \frac{h_{12,3}h_{13,2}}{\ell_{12}\ell_{13}} + \frac{h_{12,3}h_{23,1}}{\ell_{12}\ell_{23}} + \frac{h_{13,2}h_{23,1}}{\ell_{13}\ell_{23}} \\
&= \frac{h_{12,3}h_{13,2}\sin\gamma_{123}}{\ell_{12}\ell_{13}\sin\gamma_{123}} + \frac{h_{12,3}h_{23,1}\sin\gamma_{213}}{\ell_{12}\ell_{23}\sin\gamma_{213}} + \frac{h_{13,2}h_{23,1}\sin\gamma_{312}}{\ell_{13}\ell_{23}\sin\gamma_{312}} \\
&= \frac{\tilde{A}_{123}}{A_{123}}
\end{aligned}$$

where $\tilde{A}_{123}$ is the area of the "pedal triangle" whose vertices are the intersections of the altitudes through the center $C(\{1,2,3\})$ with each of the sides. The area is actually signed according to whether the values of $h_{ij,k}$ are positive or negative. It is well known that the area of a pedal triangle is the same if we move the center along a circle centered at the circumcenter and zero along the circumcircle (see, for instance, [5, Theorems 62 and 63]). Thus we see that the minor M is positive if $C(\{1,2,3\})$ is inside the circumcircle. Hence if this is true for all triangles, then the Dirichlet energy must be positive. The maximum value for M is $1/4$, realized when all the weights are zero (or equal) since in this case $C(\{1,2,3\})$ is the circumcenter.

Delaunay triangulations of closed surfaces have also been studied, for instance in [22] and later [14]. Bobenko and Springborn [2] note that Rivin's result that any Delaunay triangulation of a surface can be gotten by a sequence of flips (see [22]) implies that the full statement of Rippa's theorem applies to closed surfaces as well. The notion of weighted Delaunay extends to triangulations of a surface as well, but there is the same difficulty of using flips to find a weighted Delaunay that occurs in the planar case.

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